L^p Solutions for Stochastic Evolution Equation with Nonlinear Potential

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Abstract

This article deals with the stochastic partial differential equation

$$\begin{cases} u_t = \frac{1}{2}u_{xx} + u^\gamma \xi \\ u(0,.) = u_0 \end{cases}$$

where ξ is a space / time white noise Gaussian random field, $\gamma \in (1, +\infty)$ and u_0 is a non-negative initial condition independent of ξ satisfying

$$u_0 \geq 0, \qquad \mathbb{E}\left[\left(\int_{\mathbb{S}^1} u_0^{2\gamma}(x)dx\right)\right] < +\infty \qquad \forall \alpha \in (0,1).$$

The *space* variable is $x \in \mathbb{S}^1 = [0,1]$ with the identification $0 = 1$. The definition of the stochastic term, taken in the sense of Walsh, will be made clear in the article. The result is that there exists a non-negative solution u such that for all $\alpha \in [0,1)$,

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} u(t,x)^{2\gamma} dx dt\right)^{\alpha/2}\right] \leq C(\alpha) < +\infty.$$

where the constant $C(\alpha)$ arises in the Burkholder-Davis-Gundy inequality. The solution is unique among solutions which satisfy this. The solution is also shown to satisfy

$$\mathbb{E}\left[\int_0^T \left(\int_{\mathbb{S}^1} u(t,x)^p dx\right)^{\alpha/p} dt\right] < +\infty \qquad \forall T < +\infty, \qquad 0 < p < +\infty, \qquad \alpha \in \left(0, \frac{1}{2}\right).$$

Classifications: Primary: 60H15, Secondary: 35R60

Key words: Stochastic Partial Differential Equations

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1 Introduction

This article shows existence of solutions in suitable function spaces for the equation

$$\begin{cases} u_t = \frac{1}{2}u_{xx} + u^\gamma \xi \\ u(0,x) = u_0(x) \geq 0 \qquad \mathbb{E}\left[\left(\int_{\mathbb{S}^1} u_0^{2\gamma}(x)dx\right)^{\alpha/2}\right] < +\infty \qquad \forall \alpha \in (0,1) \qquad \gamma \in (1,\infty) \end{cases} \tag{1}$$

with space variable $x \in \mathbb{S}^1 = [0,1]$, the unit circle with identification $0 = 1$ where u_0 (the initial condition) is non-negative. Here, subscripts denote derivatives; u_t denotes the derivative of $u : \mathbb{R}_+ \times S^1 \times \Omega \to \mathbb{R}$ with respect to the first variable (the *time* variable); u_{xx} the second derivative with respect to the second variable (the *space* variable). Equation (1) is short hand for the corresponding Stochastic Integral Equation given later (after the machinery to define it has been introduced) as Equation (15); the derivatives are understood in this sense. $\xi : \mathbb{R}_+ \times \mathbb{S}^1 \times \Omega \to \mathbb{R}$ is used to denote space/time white noise and the stochastic integral in the SIE is understood in the sense of Walsh [12]. The initial condition u_0 is independent of the white noise field ξ.

Clearly there are no *strong* solutions to (1) in the sense of p.d.e.s, solutions will not be twice differentiable in the space variables or once differentiable in the time variable.

1.1 Background

Let W be a standard one dimensional Wiener process. Consider the stochastic ordinary differential equation:

$$u(t) = u_0 + \int_0^t u(s)^\gamma dW(s) \qquad u_0 \geq 0 \tag{2}$$

taken in the sense of Itô, for $\gamma > 0$. This has been well studied. Existence and behaviour of solutions can be obtained by comparison with an appropriate Bessel process, in the following way. Let $Y(t) = u^\alpha(t)$, then a minor modification of Itô's formula gives:

$$Y(t) = u_0^\alpha + \alpha \int_0^t Y(s)^{1+(\gamma-1)/\alpha} dW(s) + \frac{\alpha(\alpha-1)}{2}\int_0^t Y(s)^{1+2(\gamma-1)/\alpha} ds. \tag{3}$$

Itô's formula may be applied to $f(u(t))$ for functions $f \in C^2(\mathbb{R})$, but for $\alpha < 2$, $f(x) = |x|^\alpha$ is not twice differentiable at 0. The modification involves considering stopping times $\sigma_\epsilon = \inf\{t : u(t) < \epsilon\}$ and applying Itô's formula to $f(u(t \wedge \sigma_\epsilon))$. The comparison with Bessel processes of dimension greater than 2 in (4) will imply that $\lim_{\epsilon \to 0} \sigma_\epsilon = +\infty$ almost surely.

For $\alpha = 1 - \gamma$, where $\gamma \neq 1$,

$$Y\left(\frac{t}{(\gamma-1)^2}\right) = u_0^{1-\gamma} - (\gamma-1)W\left(\frac{t}{(\gamma-1)^2}\right) + \frac{\gamma}{2(\gamma-1)}\int_0^t \frac{1}{Y\left(\frac{r}{(\gamma-1)^2}\right)}dr$$

Now let $\widetilde{W}(t) = -(\gamma-1)W\left(\frac{t}{(\gamma-1)^2}\right)$, so that $\widetilde{W}$ is a standard Brownian motion and let $Z(t) = Y\left(\frac{t}{(\gamma-1)^2}\right)$. Then

$$Z(t) = u_0^{1-\gamma} + \widetilde{W}(t) + \frac{\left(\frac{2\gamma-1}{\gamma-1}\right) - 1}{2} \int_0^t \frac{1}{Z(s)} ds \tag{4}$$

so that Z is a $\frac{2\gamma-1}{\gamma-1}$ dimensional Bessel process. It follows that for $\gamma \neq 1$, $u^{1-\gamma}(t) = Z((\gamma-1)^2 t)$. A Bessel process of dimension greater than 2 is bounded away from 0 (see Revuz and Yor [10]). Since $\frac{2\gamma-1}{\gamma-1} > 2$ for all $\gamma > 1$, it follows that for initial condition $u_0 > 0$, the solution u is a well defined non-negative local martingale, satisfying $\sup_{0 \leq t < +\infty} u(t) < +\infty$. The following asymptotic holds:

$$\frac{u^{2(1-\gamma)}(t)}{(\gamma-1)^2 t} \longrightarrow_{(d)} Y$$

where the random variable Y has density function:

$$f(y) = \begin{cases} \frac{1}{2^{(2\gamma-1)/(2\gamma-2)}} \frac{1}{\Gamma\left(\frac{2\gamma-1}{2\gamma-2}\right)} \frac{1}{y^{1/(2\gamma-2)}} e^{-y/2} & y \geq 0 \\ 0 & y < 0. \end{cases}$$

This is a straightforward rescaling of the (unnumbered) formulae found towards the middle of p.446 of Revuz and Yor [10].

A natural question to ask is the extent to which properties of one dimensional equations are retained in the presence of mixing. For example, consider an operator A defined on functions over a countable space $\mathcal{X}$ such that $\sum_{y \in \mathcal{X}} A_{x,y} = 0$ for each $x \in \mathcal{X}$, and the system of coupled stochastic differential equations:

$$u(t,x) = u_0(x) + \int_0^t \sum_y A_{x,y} u(s,y) ds + \int_0^t u(s,x)^\gamma dW^{(x)}(s) \tag{5}$$

where $u_0(x) > 0$ for each x and $(W^{(x)})_{x \in \mathcal{X}}$ are independent Wiener processes, each with the same diffusion coefficient. How does the coupling change the nature of the system?

Now consider $\{A^{(h)}_{x,y} : x, y \in h\mathbb{Z}\}$ be defined by: $A^{(h)}_{hx,h(x+1)} = A^{(h)}_{hx,h(x-1)} = \frac{1}{2h^2}$, $A^{(h)}_{hx,hx} = -\frac{1}{h^2}$, $A^{(h)}_{x,y} = 0$ otherwise. The notation $\mathbb{E}[.]$ will be used throughout to denote 'expectation'. For each $x \in h\mathbb{Z}$, let $(W^{(h,x)})_{x \in h\mathbb{Z}}$ be independent Wiener processes satisfying $\mathbb{E}\left[W^{(h,x)}(t)\right] \equiv 0$ and $\mathbb{E}\left[W^{(h,x)}(s)W^{(h,x)}(t)\right] = (s \wedge t)\frac{1}{h}$. Note that the diffusion of the independent Wiener processes changes as $h \to 0$. Also, for $f \in C^2(\mathbb{R})$ (twice differentiable functions), $\lim_{h \to 0} A^{(h)} f = \frac{1}{2}\frac{d^2}{dx^2} f$. The operator $A^{(h)}$ is the 'discrete Laplacian' on the lattice $h\mathbb{Z}$ and its limit is the operator $\frac{1}{2}\frac{d^2}{dx^2}$ (the Laplacian on $\mathbb{R}$). Formally, the limiting equation of (5) as $h \to 0$, when $A^{(h)}$ is used in place of A and $W^{(h,.)}$ is used in place of $W^{(.)}$ is Equation (1), where ξ is space time 'white noise' and the final term of (1) is defined according to the theory of martingale measures due to Walsh [12].

Equation (1), with $\gamma > 1$, but with different conditions for the space variable, has been well studied; the main contributions are Mueller [5], Mueller and Sowers [6], Mueller [7] and Mueller [8], also in Krylov [4]. The works [5], [6] [7] and [8] consider the equation with non negative and continuous initial condition $u(0,x)$ and Dirichlet boundary conditions $u(t,0) = u(t,J) = 0$ and consider the solution for $t > 0$ and $0 \leq x \leq J$.

As with Mueller [5], approximate equations are considered, with the truncation $(u \wedge n)^\gamma$; the approximating equation is:

$$\begin{cases} u_t^{(n)} = \frac{1}{2}u_{xx}^{(n)} + (u^{(n)} \wedge n)^\gamma \xi \\ u(0,x) = u_0(x) \end{cases} \tag{6}$$

Following Theorem 2.3 of Shiga [11], Equation (6) has a unique solution, which is non-negative for n finite. Therefore, any solution to (1) obtained through approximating by (6) will be non negative. Shiga considers state space $\mathbb{R}$; the arguments for $\mathbb{S}^1$ are the same. Walsh proves existence, uniqueness, and regularity of solutions for equations similar to (6) ([12], Theorem 3.2 and Corollary 3.4). His regularity results depend on the initial condition.

In [5], existence and uniqueness of solution is shown for Equation (1) for $1 \leq \gamma < \frac{3}{2}$. Solutions to (1) agree with solutions to (6) up to time $\sigma_n = \inf\{t : \sup_x u(t,x) \geq n\}$. There is existence, uniqueness and continuity up to time $\sigma = \lim_{n\to+\infty} \sigma_n$ and then it is shown that $\mathbb{Q}(\sigma = +\infty) = 1$ for $\gamma < \frac{3}{2}$, where $\mathbb{Q}$ is used to denote the probability measure.

In [6], Mueller and Sowers study Equation (1), again with Dirichlet boundary conditions and the same conditions on the initial condition. In [6], $\gamma > \frac{3}{2}$ is considered and, with σ defined in the same way, it is shown that there exists a $\gamma_0 \geq \frac{3}{2}$ such that for $\gamma > \gamma_0$, $\mathbb{Q}(\sigma < +\infty) > 0$. The line of approach is to couple the solution to a branching process, where large peaks are regarded as particles in the branching process and offspring are peaks that are higher by some factor. It is shown that, for $\gamma > \gamma_0$, the expected number of offspring is greater than one. It follows that the branching process survives with positive probability, which corresponds to $\sigma < +\infty$. The event $\{\sigma < +\infty\}$ corresponds to the event $\{\lim_{t\uparrow\sigma} \|u(t,.)\|_\infty = +\infty\}$. In Mueller [8], the techniques of [6] are sharpened to show that for all $\gamma > \frac{3}{2}$, there is explosion of $\|u(t,.)\|_\infty$ in finite time with positive probability.

The work of Mueller and Sowers [6] and Mueller [8] shows that the L^∞ spatial norm explodes for $\gamma > \frac{3}{2}$ with positive probability, so that any technique for proving existence of solution that relies on long time existence of the L^∞ spatial norm will fail. Mueller [7] shows local existence and uniqueness for Equation (1) (with Dirichlet boundary conditions) with unbounded initial conditions, indicating that L^p solutions could exist beyond the explosion time of the L^∞ norm. Furthermore, consideration of the one dimensional SODE (2) might suggest that there is a well defined solution with long time existence of L^p norm for some $0 < p < +\infty$, since the SODE has a well defined solution with probability 1. In this article, the equation is considered on $\mathbb{S}^1$, the unit circle. That is, the space variable takes its values in $[0,1]$ where 0 and 1 are identified. Instead of taking Dirichlet boundary conditions, the identification $u(t,0) = u(t,1)$ is made and $\frac{d^2}{dx^2}$ is taken as the Laplacian on $\mathbb{S}^1$. While no comparison results are proved in this article, $\mathbb{Q}(\sigma < +\infty)$ should be greater with Dirichlet boundary conditions than on the circle. Suppose that there exists a solution to Equation (1), taken on the unit circle, with non negative intial condition satisfying $\int_{\mathbb{S}^1} u(0,x)dx = C$ for some $C > 0$. Let $U(t) = \int_{\mathbb{S}^1} u(t,x)dx$. Then $\{U(t) : t \geq 0\}$ is a non negative local martingale and, from a general result about non negative local martingales (given below), it satisfies: $\sup_{n\geq 1} n\mathbb{Q}(\sup_t U(t) > n) \leq K < +\infty$ for some K. It follows that $\int_{\mathbb{S}^1} u(t,x)dx$ is bounded almost surely in the time variable. Furthermore, the increasing process of U is simply: $\langle U \rangle(t) = \int_0^t \int_{\mathbb{S}^1} u(s,x)^{2\gamma} dxds$. Mueller and Sowers [6] followed by Mueller [8] show that there is explosion with positive probability of the L^∞ norm for $\gamma > \frac{3}{2}$. This article shows existence of solutions in appropriate L^p spaces for all $\gamma > 1$.

2 Martingale Inequalities

This section gives some basic results about non-negative continuous local martingales that will be used in the sequel. Throughout, $\mathbb{Q}$ will be used to denote the probability generic probability measure over the probability space on which the processes and random variables under discussion are defined and $\mathbb{E}$ expectation with respect to the measure $\mathbb{Q}$.

Lemma 2.1. *Let M be a non-negative continuous local martingale satisfying $M(0) = x > 0$. Let $\tau_n = \inf\{t : M(t) \geq n\}$, then*

$$\mathbb{Q}(\tau_n < \infty) \leq 1 \wedge \frac{x}{n}.$$

Proof This is well known and follows from the gambler's ruin problem. The proof is included since it is short.

Let $\tau_n = \inf\{t : M(t) = n\}$. Then τ_n is a stopping time with respect to the natural filtration of M and the stopped process $M^{(\tau_n)}$ is a martingale. It follows that, for each $n \geq 1$,

$$x = \mathbb{E}\left[M^{(\tau_n)}(t)\right] = \mathbb{E}\left[M^{(\tau_n)}(t)\mathbf{1}_{[t,+\infty)}(\tau_n)\right] + n\mathbb{Q}(t > \tau_n)$$

and since $\mathbb{E}\left[M^{(\tau_n)}(t)\mathbf{1}_{[t,+\infty)}(\tau_n)\right] \geq 0$ for all $t \geq 0$, it follows that

$$\mathbb{Q}(\tau_n < \infty) \leq 1 \wedge \frac{x}{n}$$

as required. □

Lemma 2.2. *Let M be a non-negative continuous local martingale with $M(0) = x$. Then for all $\alpha \in (0, 1)$,*

$$x^\alpha \leq \mathbb{E}\left[\sup_{0<s<\infty} M(s)^\alpha \middle| M(0) = x\right] \leq \frac{x^\alpha}{1-\alpha}.$$

Proof Again, this is well known; it is a straightforward consequence of Lemma 2.1. It is included because heavy use is made of it in the proof of the main result.

Let $\tau_n = \inf\{t : M(t) \geq n\}$, then

$$\mathbb{Q}(\tau_n < t) = \mathbb{Q}\left(\sup_{0<s<t} M(s) \geq n\right).$$

Let $\widetilde{M}$ denote the process such that $\widetilde{M}(t) = M(t)\mathbf{1}_{\{\sup_{0\leq s<+\infty} M(s)<+\infty\}}$. Then the process $\widetilde{M}$ is equivalent to M, since from Lemma 2.1, it follows that $\mathbb{Q}(\sup_{0\leq s<+\infty} M(s) < +\infty) = 1$. Let M now denote this equivalent process and set $X = \left(\sup_{0<s<+\infty} M(s)\right)$. Then, from Lemma 2.1,

$$\begin{aligned} x^\alpha &\leq \mathbb{E}[X^\alpha] = \int_0^\infty \mathbb{Q}(X^\alpha \geq y)\, dy = \int_0^\infty \mathbb{Q}\left(X \geq y^{1/\alpha}\right) dy \\ &\leq \int_0^\infty \left(1 \wedge \frac{x}{y^{1/\alpha}}\right) dy = x^\alpha + x\int_{x^\alpha}^\infty y^{-1/\alpha} dy = x^\alpha + \frac{\alpha}{1-\alpha}x^\alpha = \frac{x^\alpha}{1-\alpha} \end{aligned}$$

for all $\alpha \in (0, 1)$. □

Corollary 2.3. *Let M be a non-negative continuous local martingale. Then for all $\alpha \in (0,1)$,*

$$\mathbb{E}\left[M(0)^{\alpha}\right] \leq \mathbb{E}\left[\sup_{0<s<+\infty} M(s)^{\alpha}\right] \leq \frac{1}{1-\alpha}\mathbb{E}\left[M(0)^{\alpha}\right] \tag{7}$$

Proof Immediate. □

Lemma 2.4. *Let M be a non-negative continuous local martingale M. For $\alpha \in (0,1)$, there exists a strictly positive constant $c(\alpha)$, which does not depend on the local martingale M, such that*

$$\mathbb{E}\left[\langle M\rangle(\infty)^{\alpha/2}\right] \leq \frac{2-\alpha}{c(\alpha)(1-\alpha)}\mathbb{E}\left[M(0)^{\alpha}\right]. \tag{8}$$

Here, $c(\alpha)$ is the strictly positive constant which emerges in the usual Burkholder-Davis-Gundy inequality which states that for all local martingales N such that $N(0)=0$,

$$c(\alpha)\mathbb{E}\left[\langle N\rangle(t)^{\alpha/2}\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq t}|N(s)|^{\alpha}\right] \leq C(\alpha)\mathbb{E}\left[\langle N\rangle(t)^{\alpha/2}\right]. \tag{9}$$

Notation The constant $K(\alpha)$ will be used to denote the multiplier in Equation (8);

$$K(\alpha) = \frac{2-\alpha}{c(\alpha)(1-\alpha)}. \tag{10}$$

Proof Let $A_x = \{|\sup_{0<s<\infty} M_s - x| < x\}$ and let $\mathbf{1}_B$ denote the indicator function for a set B. Note that if $y>0$, $x>0$ and $|y-x|>x$ then $y>2x$ so that $|y-x| = y-x < y$. Using the Burkholder-Davis-Gundy inequality, it follows from Lemma 2.2 that

$$\begin{aligned}
c(\alpha)\mathbb{E}&\left[\langle M\rangle(\infty)^{\alpha/2}\middle| M(0)=x\right] \leq \mathbb{E}\left[\sup_{0<s<\infty}|M(s)-x|^{\alpha}\middle| M(0)=x\right]\\
&\leq x^{\alpha} + \mathbb{E}\left[\sup_{0<s<\infty}|M(s)-x|^{\alpha}\mathbf{1}_{A_x^c}\middle| M(0)=x\right] \leq x^{\alpha} + \mathbb{E}\left[\sup_{0<s<\infty}|M(s)|^{\alpha}\middle| M(0)=x\right]\\
&\leq x^{\alpha}\left(1+\frac{1}{1-\alpha}\right) = \frac{2-\alpha}{1-\alpha}x^{\alpha},
\end{aligned}$$

so that

$$\mathbb{E}\left[\langle M\rangle(\infty)^{\alpha/2}\right] \leq \frac{2-\alpha}{c(\alpha)(1-\alpha)}\mathbb{E}\left[M(0)^{\alpha}\right]$$

as required. □

3 Wiener Sheet, Function Spaces and Stochastic Integrals

The formal definition of the Wiener sheet (Brownian sheet) is found in Walsh [12]. It was introduced into the literature earlier by T. Kitagawa [3]. The approach taken here to the construction of a stochastic integral with respect to a Wiener sheet largely follows the approach of Walsh, with gentle modification to accommodate the situation where second moments of the stochastic integral may not exist.

An explicit statement is now given of the probability space that is necessary to define the equation under consideration.

Definition 3.1 (Wiener Sheet)**.** *Let $\mathcal{B}(A)$ denote Borel σ-field of a space A. Let $E = \mathbb{R}_+ \times \mathbb{S}^1$, $\mathcal{E} = \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{S}^1)$ the Borel σ-algebra over E and λ Lebesgue measure defined on $(E, \mathcal{E})$. A* Wiener sheet *is a random set function W defined on the sets $A \in \mathcal{E}$ of finite λ-measure such that*

1. *$W(A) \sim N(0, \lambda(A))$ for all $A \in \mathcal{E}$,*
2. *For $A, B \in \mathcal{E}$ such that $A \cap B = \phi$, $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$.*

Lemma 3.2. *The Wiener sheet is well defined.*

Proof This is proved in Walsh [12] Chapter 1 page 269. □

Definition 3.3 (Filtrations and Probability Space for the SPDE)**.** *Let $(\widetilde{\Omega}, \mathcal{F}, \mathcal{F}_t, \widetilde{\mathbb{Q}})$ denote the filtered probability space on which the Wiener sheet (Definition 3.1) is defined. That is, For $t > 0$, $\mathcal{F}_t$ is the σ-field generated by*

$$\{W([0, s], A);\ 0 \le s \le t,\ A \in \mathcal{B}(\mathbb{S}^1)\},$$

Let $(\widehat{\Omega}, \mathcal{G}_0, \widehat{\mathbb{Q}})$ denote a probability space independent of $(\widetilde{\Omega}, \mathcal{F}, \mathcal{F}_t, \widetilde{\mathbb{Q}})$ and let $u_0 : \mathbb{S}^1 \times \widehat{\Omega} \to \mathbb{R}_+$, the initial condition for (1), be measurable with respect to $\mathcal{G}_0$. Let $(\Omega, \mathcal{G}_t, \mathcal{G}, \mathbb{Q})$ be the filtered probability space, where $\Omega = \widehat{\Omega} \times \widetilde{\Omega}$, $\mathcal{G}_t = \mathcal{G}_0 \otimes \mathcal{F}_t$, $\mathbb{Q} = \widehat{\mathbb{Q}} \times \widetilde{\mathbb{Q}}$. Let $\mathcal{G} = \vee_{t \ge 0} \mathcal{G}_t$, so that $(\Omega, \mathcal{G}_t, \mathcal{G}, \mathbb{Q})$ is the probability space for (u_0, W).

Definition 3.4. *A function $f(s, x, \omega)$ is* elementary *if it is of the form: $f(s, x, \omega) = X(\omega)\mathbf{1}_{\{0\}}(s)\mathbf{1}_A(x)$ where X is $\mathcal{G}_0$ measurable or, for $0 \le a \le b$,*

$$f(s, x, \omega) = X(\omega)\mathbf{1}_{(a,b]}(s)\mathbf{1}_A(x)$$

where X is bounded and $\mathcal{G}_a$ measurable and $A \in \mathcal{B}(\mathbb{S}^1)$. f is simple *if it is the sum of elementary functions. The class of simple functions will be denoted by $\mathcal{S}$.*

Definition 3.5. *The* predictable *σ-field $\mathcal{P}$ is the σ-field generated by $\mathcal{S}$. A function is* predictable *if it is $\mathcal{P}$-measurable.*

Definition 3.6 (Function Spaces)**.** *Let $g \in \mathcal{P}$. For $\alpha \in (0, 2]$, the following function spaces will be employed, with $p > 1$ (mostly $p = 2\gamma$):*

$$\begin{cases} \mathcal{S}_{p,\alpha;K} = \left\{ g : g \in \mathcal{P} : \|g\|_{p,\alpha} := \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} |g(s,x)|^p dx ds \right)^{\alpha/2} \right]^{1/p} < K \right\} \\ \mathcal{S}_{p,\alpha} = \cup_{K > 1} \mathcal{S}_{p,\alpha;K} \end{cases} \tag{11}$$

The space $\mathcal{S}_{p,\alpha}$ is equipped with the metric $d_{p,\alpha}$ defined by:

$$d_{p,\alpha}(g, h) = \|g - h\|_{p,\alpha} \tag{12}$$

for $0 < \alpha < 2$. Two functions $g, h \in \mathcal{P}$ are said to be (p, α) equivalent if and only if $d_{p,\alpha}(g, h) = 0$.

Note Note that $\|.\|_{p,\alpha}$ is not a norm, since it satisfies $\|cf\|_{p,\alpha} = |c|^{\alpha/2}\|f\|_{p,\alpha}$, which does not equal $|c|\|f\|_{p,\alpha}$ unless $\alpha = 2$. The distance $d_{p,\alpha}$ is a metric for all $\alpha \le 2$. For $\alpha = 2$, $p > 1$, $d_{p,2}$ is clearly a metric. This will be used for solutions to approximating equations whose moments are all well defined. For $\alpha \in (0,2)$, the following lemma shows that $d_{p,\alpha}$ is a metric.

Lemma 3.7. *For $p \ge 2$, the quantity $d_{p,\alpha}$ defined in Equation (12) is a metric for $\alpha \in (0,2)$, in the sense that*

1. *The triangle inequality holds; for any $f, g, h \in \mathcal{S}_{p,\alpha}$,*

$$d_{p,\alpha}(f,g) \le d_{p,\alpha}(f,h) + d_{p,\alpha}(h,g)$$

2. *$d_{p,\alpha}(f,g) = 0$ implies that $f = g$ up to α equivalence.*

3. *$d_{p,\alpha}(f,g) = d_{p,\alpha}(g,f)$.*

Proof of Lemma 3.7 For the second point, equivalence class is *defined* such that f and g are in the same (p,α) equivalence class if and only if $d_{p,\alpha}(f,g) = 0$. The third point is clear. It only remains to prove the triangle inequality. Let $b_1 = \left(\int_0^\infty \int_{\mathbb{S}^1} |f-g|^p(t,x)dxdt\right)^{1/p}$ and $b_2 = \left(\int_0^\infty \int_{\mathbb{S}^1} |g-h|^p(t,x)dxdt\right)^{1/p}$. Then, using $\|f+g\| \le \|f\| + \|g\|$ for L^p norms and that $(a_1 + \ldots + a_n)^{\alpha/2} \le a_1^{\alpha/2} + \ldots + a_n^{\alpha/2}$ for nonnegative $a_1, \ldots, a_n$ and $0 < \alpha < 2$, together with Hölder's inequality:

$$\begin{aligned}
d_{p,\alpha}(f,h) &= \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} |(f-g)+(g-h)(t,x)|^p dxdt\right)^{\alpha/2}\right]^{1/p} \\
&\le \mathbb{E}\left[\left(\left(\int_0^\infty \int_{\mathbb{S}^1} |f-g|^p dxdt\right)^{1/p} + \left(\int_0^\infty \int_{\mathbb{S}^1} |g-h|^p dxdt\right)^{1/p}\right)^{p\alpha/2}\right]^{1/p} \\
&\le \mathbb{E}\left[\left(\left(\int_0^\infty \int_{\mathbb{S}^1} |f-g|^p dxdt\right)^{\alpha/2p} + \left(\int_0^\infty \int_{\mathbb{S}^1} |g-h|^p dxdt\right)^{\alpha/2p}\right)^{p}\right]^{1/p} \\
&\le \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} |f-g|^p dxdt\right)^{\alpha/2}\right]^{1/p} + \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} |g-h|^p dxdt\right)^{\alpha/2}\right]^{1/p} \\
&= d_{p,\alpha}(f,g) + d_{p,\alpha}(g,h)
\end{aligned}$$

The third to fourth line follows using: for non-negative A and B, $\mathbb{E}[(A+B)^p]^{1/p} \le \mathbb{E}[A^p]^{1/p} + \mathbb{E}[B^p]^{1/p}$. $\square$

Lemma 3.8. *For $p > 1$ and $\alpha \in (0,1)$, the space $\mathcal{S}_{p,\alpha}$, equipped with metric $d_{p,\alpha}$ is complete.*

Proof Consider a Cauchy sequence $(u^{(n)})_{n\ge 0}$ in the space $\mathcal{S}_{p,\alpha}$. There is a subsequence $(u^{(n_k)})_{k\ge 1}$ such that $d_{p,\alpha}(u^{(n_k)}, u^{(n_{k+1})}) \le e^{-k}$. Let

$$G = |u^{(n_0)}| + \lim_{N\to+\infty} \sum_{k=1}^{N} |u^{(n_k)} - u^{(n_{k-1}))}|,$$

The limit is pointwise well defined $\lambda_{\mathbb{R}_+} \otimes \lambda_{\mathbb{S}^1} \otimes \mathbb{Q}$ - almost surely, where $\lambda_{\mathbb{R}_+}$ and $\lambda_{\mathbb{S}^1}$ denote Lebesgue measure over the time and spatial variables respectively. This is seen as follows:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} \left(|u^{(n_0)}| + \sum_{k=1}^\infty |u^{(n_k)} - u^{(n_{k-1})}|\right)^p dxds\right)^{\alpha/2}\right]^{1/p} \\
&\quad \leq \mathbb{E}\left[\left(\left(\int_0^\infty \int_{\mathbb{S}^1} |u^{(n_0)}|^p dxdt\right)^{1/p} + \sum_{k=1}^\infty \left(\int_0^\infty \int_{\mathbb{S}^1} |u^{(n_k)} - u^{(n_{k-1})}|^p dxdt\right)^{1/p}\right)^{p\alpha/2}\right]^{1/p} \\
&\quad \leq \mathbb{E}\left[\left(\left(\int_0^\infty \int_{\mathbb{S}^1} |u^{(n_0)}|^p dxdt\right)^{\alpha/2p} + \sum_{k=1}^\infty \left(\int_0^\infty \int_{\mathbb{S}^1} |u^{(n_k)} - u^{(n_{k-1})}|^p dxdt\right)^{\alpha/2p}\right)^{p}\right]^{1/p} \\
&\quad \leq \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} |u^{(n_0)}|^p dxdt\right)^{\alpha/2}\right]^{1/p} + \sum_{k=1}^\infty \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} |u^{(n_k)} - u^{(n_{k-1})}|^p dxdt\right)^{\alpha/2}\right]^{1/p} \\
&= \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} |u^{(n_0)}|^p dxdt\right)^{\alpha/2}\right]^{1/p} + \sum_{k=1}^\infty d_{p,\alpha}(u^{(n_{k-1})}, u^{(n_k)}) < +\infty.
\end{aligned}$$

It follows that G is well defined almost surely and hence that $u^{(n_0)} + \sum_{k=1}^N (u^{(n_k)} - u^{(n_{k-1})})$ converges pointwise almost surely to a limit u such that $|u| \leq G$. Now, from the dominated convergence theorem,

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} |u^{(n_k)} - u| dxdt\right)^{\alpha/2}\right] \stackrel{k\to+\infty}{\longrightarrow} 0.$$

Hence the space is complete. □

Now the stochastic integral with respect to the Wiener sheet may be constructed.

Remark Although the construction is essentially the same as Walsh [12], the stochastic integral here is constructed over the whole time range $[0, \infty)$. The functions of interest (solutions to Equation (1)) decay as $t \to +\infty$ and the definition presents no difficulty.

Let $\mathcal{C}$ denote the class of functions $g \in \mathcal{P}$ which satisfy the following:

- there is an $m_0 < +\infty$, disjoint sets $\{B_j \quad j = 1, \ldots, m_0\}$ and $\mathcal{G}_0$-measurable random variables $f_1, \ldots, f_{m_0}$,
- a collection $0 = t_0 < t_1 < \ldots < t_n < +\infty$
- disjoint sets $\{A_{i,j} \quad i = 1, \ldots, n; j = 1, \ldots, m_i\}$ where $m_1, \ldots, m_n < +\infty$ and for each $i \in \{0, 1, \ldots, n-1\}$, $\cup_{j=1}^{m_i} A_{i,j} = \mathbb{S}^1$
- a collection $(g_{i,j} : i \in \{0, \ldots, n-1\}, j \in \{1, \ldots, m_i\})$ of random variables such that $g_{i,j}$ is $\mathcal{G}_{t_i}$ measurable for each $j \in \{1, \ldots, m_i\}$

and g is given by:

$$g(s,x,\omega)=\sum_{j=1}^{m_0} f_j(\omega)\mathbf{1}_{\{0\}}(s)\mathbf{1}_{B_j}(x)+\sum_{i=0}^{n-1}\sum_{j=1}^{m_i} g_{i,j}(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(s)\mathbf{1}_{A_{i,j}}(x)$$

and satisfies:

$$\mathbb{E}\left[\left(\int_0^\infty\int_{\mathbb{S}^1} g(t,x)^2 dxdt\right)^{\alpha/2}\right]=\mathbb{E}\left[\left(\sum_{i=1}^{n}(t_{i+1}-t_i)\sum_{j=1}^{m_i}|A_{ij}|g_{ij}^2\right)^{\alpha/2}\right]<+\infty.$$

For $g\in\mathcal{C}$, the stochastic integral is defined as:

$$I(g)(t)=\sum_{i=0}^{n-1}\sum_{j=1}^{m_i} g_{i,j}W((t\wedge t_i,t\wedge t_{i+1}],A_{ij}).$$

It is clear that, for $g\in\mathcal{C}$, the stochastic integral $I(g)$ is a continuous local martingale with quadratic variation given by:

$$\langle I(g)\rangle(t)=\int_0^t\int_{\mathbb{S}^1} g^2(s,x)dxds.$$

Lemma 3.9. *Let $\alpha\in(0,1)$. The space of continuous local martingales $M:\mathbb{E}\left[\sup_t|M(t)|^\alpha\right]<+\infty$ with metric $D_\alpha(M,N)=\mathbb{E}\left[\sup_t|M(t)-N(t)|^\alpha\right]$ is complete in the following sense; let $M^{(n)}$ denote a sequence of local martingales satisfying $\sup_n\mathbb{E}\left[\sup_t|M^{(n)}(t)|^\alpha\right]<K$ for some $K<+\infty$ such that*

$$\lim_{n\to+\infty}\sup_{N\geq n}\mathbb{E}\left[\sup_t\left|M^{(N)}(t)-M^{(n)}(t)\right|^\alpha\right]=0,\tag{13}$$

then there is a continuous local martingale M satisfying $\mathbb{E}\left[\sup_t|M(t)|^\alpha\right]<K$ such that

$$\lim_{n\to+\infty}\mathbb{E}\left[\sup_t|M^{(n)}(t)-M(t)|^\alpha\right]=0.$$

Proof of Lemma 3.9 Firstly, for completeness, consider a subsequence $M^{(n_k)}$ such that

$$\lim_{m\to+\infty}\mathbb{E}\left[\left(\sum_{k=1}^m\sup_{0<t<+\infty}|M^{(n_k)}(t)-M^{(n_{k-1})}(t)|\right)^\alpha\right]<+\infty.$$

Such a subsequence exists, by hypothesis, since

$$\left(\sum_{k=1}^m\sup_{0<t<+\infty}|M^{(n_k)}(t)-M^{(n_{k-1})}(t)|\right)^\alpha\leq\sum_{k=1}^m\sup_{0<t<+\infty}|M^{(n_k)}(t)-M^{(n_{k-1})}(t)|^\alpha.$$

and, by hypothesis (13), a subsequence $(M^{(n_k)})_{k\geq1}$ can be extracted such that

$$\sum_{k=1}^\infty\mathbb{E}\left[\sup_{0<t<+\infty}|M^{(n_k)}(t)-M^{(n_{k-1})}(t)|^\alpha\right]<+\infty.$$

Let $G=\sup_t M^{(n_0)}(t)+\sum_{k=1}^\infty\sup_t|M^{(n_k)}(t)-M^{(n_{k-1})}(t)|$, then $\mathbb{E}\left[G^\alpha\right]<+\infty$ so that $G<+\infty$ almost surely and hence $M=M^{(n_0)}+\sum_{k=1}^\infty(M^{(n_k)}-M^{(n_{k-1})})$ exists almost surely and converges

almost surely uniformly in t. In particular, since each $M^{(n_k)}$ is continuous almost surely, the following argument gives that M is continuous almost surely. It is necessary and sufficient to show, for each $T < +\infty$, for all $\epsilon > 0$ there exists a $\delta(T, \omega, \epsilon) > 0$ almost surely (suppressing notation, written $\delta(\epsilon)$), such that

$$\sup_{0 \leq t \leq T} \sup_{|t-s| < \delta(\epsilon)} |M(t) - M(s)| < \epsilon. \tag{14}$$

Now, for arbitrary ϵ, there exists a $k > 0$ such that $\sup_t |M(t) - M^{(n_k)}(t)| < \frac{\epsilon}{3}$ and, for such n_k there exists a $\delta(\epsilon)$ such that $\sup_{|t-s|<\delta(\epsilon))} |M^{(n_k)}(t) - M^{(n_k)}(s)| < \frac{\epsilon}{3}$. Using this $\delta(\epsilon)$, it follows that (14) holds.

Finally, the local martingale property is established. For a fixed $\alpha \in (0, 1)$, choose a sequence $M^{(n_j)}$ such that, for each j,

$$\mathbb{E}\left[\sup_t |M^{(n_j)}(t) - M(t)|^\alpha\right] < \frac{1}{j^2}.$$

Let

$$\tau_N = \inf\{t : |M(t)| > N\}, \qquad \tau_N^{(j)} = \inf\{t : |M^{(n_j)}(t)| > N\}$$

then (clearly) $\lim_{N \to +\infty} \tau_N = +\infty$ almost surely. For $t > s > 0$,

$$\begin{aligned} \mathbb{E}\left[M(t \wedge \tau_N) | \mathcal{F}_s\right] &= \mathbb{E}\left[M^{(n_j)}(t \wedge \tau_N^{(j)}) | \mathcal{F}_s\right] + \mathbb{E}\left[M(t \wedge \tau_N) - M^{(n_j)}(t \wedge \tau_N^{(j)}) | \mathcal{F}_s\right] \\ &= M^{(n_j)}(s \wedge \tau_N^{(j)}) + \mathbb{E}\left[M(t \wedge \tau_N) - M^{(n_j)}(t \wedge \tau_N^{(j)}) | \mathcal{F}_s\right]. \end{aligned}$$

Now, $M^{(n_j)}(t \wedge \tau_N^{(j)}) \stackrel{j \to +\infty}{\longrightarrow} M(t \wedge \tau_N)$ almost surely for all $t > 0$. Since $|M(t \wedge \tau_N) - M^{(n_j)}(t \wedge \tau_N^{(j)})| < N$, it follows from the dominated convergence theorem that the second term converges to 0 as $j \to +\infty$ and hence

$$\mathbb{E}\left[M(t \wedge \tau_N) | \mathcal{F}_s\right] = M(s \wedge \tau_N)$$

hence M is a continuous local martingale. □

For $g \in \mathcal{S}_{2,\alpha}$, the stochastic integral may now be constructed without delay. If $\|g\|_{2,\alpha} = K$, consider an approximating sequence of functions $g^{(n)} \in \mathcal{C}$ such that $\|g^{(n)}\|_{2,\alpha} \leq 2K$ for each n and such that $\lim_{n \to +\infty} \|g^{(n)} - g\|_{2,\alpha} = 0$. The stochastic integral $I(g)$ is defined as the limit of $I(g^{(n)})$, in the sense of the convergence of local martingales of Lemma 3.9. It follows from the Burkholder-Davis-Gundy inequality that for $\alpha < 1$,

$$\mathbb{E}\left[\sup_t \left|I(g^{(n)})(t) - I(g)(t)\right|^\alpha\right] \leq C(\alpha) \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} (g^{(n)} - g)^2(t, x) dx dt\right)^{\alpha/2}\right] \stackrel{n \to +\infty}{\longrightarrow} 0$$

where $C(\alpha)$ is the universal constant from the Burkholder-Davis-Gundy inequality.

4 Definition and Existence of Solution

Equation (1) is understood as the equivalent Stochastic Integral Equation (SIE) given by (15):

$$u(t,x) = P_t u_0(x) + \int_0^t \int_{\mathbb{S}^1} p_{t-r}(x-y) u^\gamma(r,y) W(dr,dy) \qquad \mathbb{Q}-\text{a.s.} \quad 0 \le t < +\infty, \quad x \in \mathbb{S}^1. \quad (15)$$

where $p : [0,+\infty) \times \mathbb{S}^1 \to \mathbb{R}_+$ satisfies

$$\begin{cases} p_t = \frac{1}{2} p_{xx} \\ p(0,.) = \delta_0(.) \end{cases}$$

δ_0 denotes the Dirac delta function, with unit mass at 0 and $P_t f(x) = \int_{\mathbb{S}^1} p_t(x-y) f(y) dy$.

For the *state space*, we define a solution to (15) as a non-negative function u which satisfies (15) and such that for any non-negative $\phi \in C^2(\mathbb{S}^1)$ and any $T \in \mathbb{R}$, $\sup_{0 \le t \le T} \int_{\mathbb{S}^1} u(t,x)\phi(x)dx < +\infty$ $\mathbb{Q}$-almost surely and for any $T \in \mathbb{R}_+$, $\int_0^T \int_{\mathbb{S}^1} u^{2\gamma}(t,x) dx dt < +\infty$ $\mathbb{Q}$-almost surely.

A priori, if there is a well defined non-negative solution, then $U(t) := \int_{\mathbb{S}^1} u(t,x)dx$ will be a non-negative local martingale and hence $\mathbb{E}\left[\sup_{0<t<+\infty} U(t)^\alpha\right] < +\infty$ for all $\alpha \in (0,1)$. Furthermore, if there is a well defined solution, then the increasing process of this non-negative local martingale is: $\langle U \rangle(t) = \int_0^t u^{2\gamma}(s,x) dx ds$ and will therefore (by the Burkholder Davis Gundy inequality) satisfy $\mathbb{E}\left[\langle U \rangle(+\infty)^{\alpha/2}\right] < +\infty$ for all $\alpha \in (0,1)$. Hence these conditions are not restrictive.

Equation (15) is the *mild form* of Equation (1); a function u that satisfies (15) is known as a *mild solution* to (1).

Existence of solution is established by considering suitable approximating sequences $(u^{(n)})_{n \ge 1}$ where, for each n, $u^{(n)}$ exists and showing that there is a function u and a subsequence $u^{(n_j)}$ such that for $\alpha \in (0,1)$, $d_{2\gamma,\alpha}(u^{(n_j)}, u) \stackrel{j \to +\infty}{\longrightarrow} 0$, which satisfies Equation (1). The following approximating sequence is used: for $n \ge 1$, the function $u^{(n)}$ is the solution to Equation (16) below:

$$u_t^{(n)}(t,x) = P_t u_0(x) + \int_0^t \int_{\mathbb{S}^1} p_{t-s}(x-y)(u^{(n)}(s,y) \wedge n)^\gamma W(ds,dx) \quad (16)$$

Notation Let $U_0 = \int_{\mathbb{S}^1} u_0(x) dx$.

Lemma 4.1. *Recall that $\gamma > 1$. For each $n \ge 1$, there exists a unique solution to Equation (16) in $\mathcal{S}_{2\gamma,2}$ equipped with metric $d_{2\gamma,2}$. For all $\alpha \in (0,1)$, there is a constant*

$$\widetilde{K}(\alpha) := K(\alpha) \mathbb{E}\left[U_0^\alpha\right] < +\infty \quad (17)$$

where $K(\alpha) > 0$ is from Equation (10) such that:

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} (u^{(n)}(s,y) \wedge n)^{2\gamma} dy ds\right)^{\alpha/2}\right] < \widetilde{K}(\alpha) < +\infty. \quad (18)$$

This constant $\widetilde{K}(\alpha)$ does not depend on n.

Proof of Lemma 4.1 The first statement follows similarly to Walsh [12] Theorem 3.2 page 313. Only minor modifications are required to deal with space variable in $\mathbb{S}^1$ rather than $[0, L]$ with Dirichlet boundary conditions and these are omitted. Walsh restricted his construction to finite time intervals $[0, T]$ where $T < +\infty$ and did not consider the whole real line $[0, +\infty)$. Let $u^{(n,T)}$ denote the function that provides the unique solution up to time T and $u^{(n,T)}(t, .) \equiv 0$ for all $t > T$ and let $u^{(n)} = \lim_{T\to+\infty} u^{(n,T)}$. Then $u^{(n)}$ is well defined and provides the unique solution up to time T for all $T \in \mathbb{R}_+$.

To prove the second statement, let $U^{(n)}(t) = \int_{\mathbb{S}^1} u^{(n)}(t, x)dx$ and note that $U^{(n)}$ is a non negative martingale that satisfies

$$U^{(n)}(t) = U_0 + \int_0^t \int_{\mathbb{S}^1} (u^{(n)}(s, y) \wedge n)^\gamma W(dy, ds).$$

It is straightforward that for finite n, $U^{(n)}$ is a martingale, since it is a stochastic integral in the sense of Walsh with bounded integrand. Its increasing process is:

$$\langle U^{(n)}\rangle(t) = \int_0^t \int_{\mathbb{S}^1} (u^{(n)}(s, y) \wedge n)^{2\gamma} dyds.$$

It follows from Lemma 2.4 that for all $\alpha < 1$,

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} (u^{(n)}(s, y) \wedge n)^{2\gamma} dyds\right)^{\alpha/2}\right] \le \widetilde{K}(\alpha)$$

thus proving the second statement of Lemma 4.1. For the first statement, $u^{(n)}$ is well defined and satisfies the equation. $\square$

Lemma 4.2 below, showing that there is a subsequence of $u^{(n)}$ such that $u^{(n)}$ is Cauchy in $\mathcal{S}_{2\gamma,\alpha}$ and $u^{(n)\gamma}$ is Cauchy in $\mathcal{S}_{2,\alpha}$ for $\alpha \in (0, 1)$, is the crucial result for establishing existence of solution to Equation (1); it is then a simple corollary of the lemma that the terms in Equation (6) corresponding to the subsequence converge to the corresponding terms in Equation (1).

The proof of Lemma 4.2 is substantial and requires several parts. Firstly, a weakly convergent subsequence of $u^{(n)}$ is established, with a corresponding limit u. From this, convergence of the corresponding local martingales $U^{(n_j)}$ defined by $U^{(n_j)}(t) = \int_{\mathbb{S}^1} u^{(n_j)}(t, x)dx$ is established, where $(u^{(n_j)})_{j\ge1}$ is a weakly convergent subsequence. The crucial point for proceeding from this weak convergence to showing convergence of $u^{(n_j)\gamma}$ using the metric $d_{2,\alpha}$ is consideration of the quadratic variation of $U^{(n_j)} - U^{(n_k)}$, $k \ge j$, $j \to +\infty$. Since $U^{(n_j)} - U^{(n_k)}$ converges to 0, the Burkholder-Davis-Gundy inequalities give convergence of quadratic variation to 0, thus giving convergence in norm, hence convergence of the martingale term in Equation (6) along the subsequence.

Lemma 4.2. *Let $v^{(n)} = n \wedge u^{(n)}$. There exists a subsequence $(n_k)_{k\ge1}$, a function $u \in \mathcal{S}_{2\gamma,\alpha}$ for all $\alpha < 1$ and a sequence $\widetilde{u}^{(n_k)}$ such that $\widetilde{u}^{(n_k)} \stackrel{(d)}{=} u^{(n_k)}$ for all $k \ge 1$. Let $\widetilde{v}^{(n)} = n \wedge \widetilde{u}^{(n)}$, then for all $\alpha < 1$*

$$\lim_{j\to+\infty} d_{2\gamma,\alpha}(\widetilde{v}^{(n_j)}, u) = 0$$

and

$$\lim_{j\to+\infty} d_{2,\alpha}(\widetilde{v}^{(n_j)\gamma}, u^\gamma) = 0.$$

Proof Let $U_0 = \int_{\mathbb{S}^1} u_0(x)dx$, the total mass of the initial condition. From Corollary 2.3 and Lemma 4.1, it follows that for all $\alpha < 1$,

$$\begin{cases} \mathbb{E}\left[\sup_{0\le t<+\infty} U^{(n)\alpha}(t)\right] \le \frac{1}{1-\alpha}\mathbb{E}\left[U_0^\alpha\right] \\ \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} v^{(n)}(s,y)^{2\gamma} dyds\right)^{\alpha/2}\right] \le \widetilde{K}(\alpha) \end{cases} \tag{19}$$

where $\widetilde{K}(\alpha)$ is defined in (17). Consider $\mathbb{R}_+ \times \mathbb{S}^1$ endowed with the metric d defined as:

$$d((s,x),(t,y)) = \sqrt{(x-y)^2 + \left(\frac{1}{1+s} - \frac{1}{1+t}\right)^2}.$$

The space $\mathbb{R}_+ \cup \{+\infty\} \times \mathbb{S}^1$ with the metric d is compact. Now consider the space of measures over $\mathbb{R}_+ \times \mathbb{S}^1$ defined by:

$$\mathcal{W} = \left\{ w(t,dx)dt, \quad w : \mathbb{R}_+ \cup \{+\infty\} \times \mathcal{B}(\mathbb{S}^1) \to \mathbb{R}_+ : \|w\| := \sup_{0<t<+\infty} w(t,\mathbb{S}^1) < +\infty \right\}$$

For $A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{S}^1)$, set (with slight abuse of notation, where the meaning is clear):

$$w(A) := \int_0^\infty \frac{1}{(1+t)^2} \int_{\mathbb{S}^1} \mathbf{1}_A(t,x) w(t,dx)dt$$

and equip $\mathcal{W}$ with the Prohorov metric:

$$\rho(v,w) = \inf\left\{\alpha > 0 : v(A) \le w(A_\alpha) + \alpha \quad \text{and} \quad w(A) \le v(A_\alpha) + \alpha \quad \forall A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathbb{B}(\mathbb{S}^1)\right\} \tag{20}$$

where $A_\alpha = \left\{(t,x) \in \mathbb{R}_+ \times \mathbb{S}^1 : \inf_{(s,y)\in A} d((t,x),(s,y)) \le \alpha\right\}$.

Consider a sequence $w^{(n)}$ in $\mathcal{W}$ such that $\sup_n \|w^{(n)}\| \le N$; in other words, $\sup_n \sup_t w^{(n)}(t,\mathbb{S}^1) \le N$. There exists a constant $K \in [0,N]$ and a subsequence $(n_j)_{j\ge 0}$ such that

$$\int_0^\infty \frac{1}{(1+t)^2} w^{(n_j)}(t,\mathbb{S}^1)dt \stackrel{j\to+\infty}{\longrightarrow} K.$$

If $K = 0$, then $w(t,dx) \equiv 0$ is the limit. Otherwise, let $K_j = w^{(n_j)}(\mathbb{R}_+ \times \mathbb{S}^1)$ and note that the sequence of measures defined over $\mathcal{B}(\mathbb{R}_+ \times \mathbb{S}^1)$ by $\mu_j(A) = \frac{1}{K_j} w^{(n_j)}(A)$ is a sequence of *probability* measures. It is a standard result that if (X,d) is a compact metric space, then the space of probability measures over X with associated Prohorov metric, denoted $(\mathcal{P}(X),\rho)$ is weakly compact. Therefore the sequence μ_j has a weakly convergent subsequence μ_{j_k} with a weak limit μ and hence the sequence $w^{(n)}$ has a convergent subsequence $w^{(n_{j_k})}$ with limit $w = K\mu$ so that, with this metric, the sets $\{w \in \mathcal{W} : \|w\| \le N\}$ are compact for each $N < +\infty$.

Now consider the sequence $w^{(n)}(t,dx) = u^{(n)}(t,x)dx$. Let $w'(t,x)$ denote the density, if it exists, such that $w(t,dx) = w'(t,x)dx$. With abuse of notation, where the meaning is clear, $\rho(v',w')$ will be written for $\rho(v,w)$ and $\|w'\|$ for $\|w\|$. Now consider the sequence $u^{(n)}$. Using $U^{(n)}(t) = \int_{\mathbb{S}^1} u^{(n)}(t,x)dx$, it follows by Markov's inequality that for any $\alpha \in (0,1)$,

$$\begin{aligned}\mathbb{Q}\left(\|u^{(n)}\| > M\right) &= \mathbb{Q}\left(\sup_{0\le t<+\infty} U^{(n)}(t) > M\right)\\ &\le \frac{1}{M^\alpha}\mathbb{E}\left[\sup_{0\le t<+\infty} U^{(n)\alpha}(t)\right] \le \frac{1}{M^\alpha(1-\alpha)}\mathbb{E}\left[U_0^\alpha\right] < +\infty\end{aligned}$$

by Corollary 2.3. It follows that

$$\lim_{M\to+\infty}\sup_{n\ge1}\mathbb{Q}\left(\|u^{(n)}\| > M\right) = 0$$

and hence tightness follows.

From Prohorov's theorem (Kallenberg [2] page 309 Theorem 16.3), stating that tightness implies relative compactness, it follows that there exists a subsequence $(u^{(n_j)})_{j\ge1}$ and a limit u such that for all bounded continuous f

$$\lim_{j\to+\infty}\mathbb{E}\left[f(u^{(n_j)})\right] = \mathbb{E}\left[f(u)\right].$$

Here 'bounded and continuous' means bounded and continuous functions over the space, along with its metric, on which u is defined; namely $\mathcal{S}_{2\gamma,\alpha}$ with metric $d_{2\gamma,\alpha}$ for any $\alpha \in (0,1)$.

By the Skorohod Representation Theorem (see, for example Theorem 4.30, page 79 of Kallenberg [2]), there exists a sequence $(\widetilde{u}^{(n_j)})_{j\ge1}$ and a limit $\widetilde{u}$ where for each j, $u^{(n_j)} \stackrel{(d)}{=} \widetilde{u}^{(n_j)}$ and $u \stackrel{(d)}{=} \widetilde{u}$ and such that $\rho(\widetilde{u}^{(n_j)},\widetilde{u}) \to 0$ almost surely, where ρ is the metric defined by Equation (20) (writing $\rho(w',v')$ for $\rho(w,v)$).

From now on, this representation will be used; $u^{(n_j)}$ will be used to denote $\widetilde{u}^{(n_j)}$ and u to denote $\widetilde{u}$.

Since the elements of the sequence $(u^{(n_j)})_{j\ge1}$ are defined on the same probability space, it follows that

$$u^{(n_j)}(t,x) = u(0,x) + \int_0^t\int_{\mathbb{S}^1} p(t-s,x-y)(u^{(n_j)}(s,y)\wedge n_j)^\gamma W^{(n_j)}(ds,dy)$$

where for each j, $W^{(n_j)}$ is a Wiener sheet such that for each $A \in \mathcal{B}(\mathbb{S}^1)$, $W^{(n_j)}(t,A)$ is measurable with respect to $\mathcal{F}_t$ (Definition 3.3). It follows that $u^{(n_j)}$ satisfies:

$$u^{(n_j)}(t,x) = u(0,x) + \int_0^t\int_{\mathbb{S}^1} p(t-s,x-y)\sigma_j(s,y)(u^{(n_j)}(s,y)\wedge n_j)^\gamma W(ds,dy) \tag{21}$$

where σ_j is a measurable function adapted to $(\mathcal{G}_t)_{t\ge0}$ such that for each $(s,y) \in \mathbb{R}_+ \times \mathbb{S}^1$, $\sigma_j(s,y) = 1$ or -1.

Let $U(t) = \int_{\mathbb{S}^1} u(t,x)dx$ and $U^{(n)}(t) = \int_{\mathbb{S}^1} u^{(n)}(t,x)dx$. By taking sets $\widetilde{A} = A \times \mathbb{S}^1$, it follows from the definition of the Prohorov-style metric ρ that

$$\rho(u^{(n_j)}, u^{(n_k)}) \geq \inf \left\{ \alpha > 0 : \int_A \frac{1}{(1+t)^2} \left(U^{(n_j)}(t) - U^{(n_k)}(t) \right) dt \leq \alpha + \int_{A_\alpha \setminus A} \frac{1}{(1+t)^2} U^{(n_k)}(t) dt \right.$$
$$\left. \text{and} \quad \int_A \frac{1}{(1+t)^2} \left(U^{(n_k)}(t) - U^{(n_j)}(t) \right) dt \leq \alpha + \int_{A_\alpha \setminus A} \frac{1}{(1+t)^2} U^{(n_j)}(t) dt \quad \forall A \in \mathcal{B}(\mathbb{R}_+) \right\}$$

where

$$A_\alpha = \left\{ t \in \mathbb{R}_+ : \inf_{s \in A} \sqrt{\left(\frac{1}{1+s} - \frac{1}{1+t} \right)^2} \leq \alpha \right\}. \tag{22}$$

Let $\alpha_{jk} = \alpha_{kj} = \rho(u^{(n_j)}, u^{(n_k)})$. Then $\lim_{j \to +\infty} \left(\lim_{k \to +\infty} \rho(u^{(n_j)}, u^{(n_k)}) \right) = 0$ almost surely. It follows from the definition that for all $A \in \mathcal{B}(\mathbb{R}_+)$:

$$\begin{aligned} & -\alpha_{jk} - \int_{A_{\alpha_{jk}} \setminus A} \frac{1}{(1+t)^2} U^{(n_j)}(t) dt \\ & \leq \int_A \frac{1}{(1+t)^2} \left(U^{(n_j)}(t) - U^{(n_k)}(t) \right) dt \leq \alpha_{jk} + \int_{A_{\alpha_{jk}} \setminus A} \frac{1}{(1+t)^2} U^{(n_k)}(t) dt. \end{aligned}$$

Establishing compactness under the Prohorov metric For a set A, the set A_α was defined by (22). The same notation will be used for sets $A^{(j,k;\epsilon)}$, $\widetilde{A}^{(j,k;\epsilon)}$, $C^{(j,k;\epsilon)}$ and $\widetilde{C}^{(j,k;\epsilon)}$ below. For example, $C^{(j,k;\epsilon)}_{\alpha_{jk}}$ is defined as the set $\left\{ t \in \mathbb{R}_+ : \inf_{s \in C^{(j,k;\epsilon)}} \sqrt{\left(\frac{1}{1+s} - \frac{1}{1+t} \right)^2} \leq \alpha_{jk} \right\}$.

The next task is to show that if $\lim_{j,k \to +\infty} \alpha_{j,k} = 0$, $\mathbb{Q}$ almost surely, then

$$\lim_{j,k \to +\infty} \int_{C^{(j,k;\epsilon)}_{\alpha_{jk}} \setminus C^{(j,k;\epsilon)}} \frac{1}{(1+t)^2} U^{(n_j)}(t) dt = 0 \quad \text{and} \quad \lim_{j,k \to +\infty} \int_{C^{(j,k;\epsilon)}_{\alpha_{jk}} \setminus C^{(j,k;\epsilon)}} \frac{1}{(1+t)^2} U^{(n_k)}(t) dt = 0$$

both for $C^{(j,k;\epsilon)} = A^{(j,k;\epsilon)}$ and $C^{(j,k;\epsilon)} = \widetilde{A}^{(j,k;\epsilon)}$, where $A^{(j,k;\epsilon)}$ and $\widetilde{A}^{(j,k;\epsilon)}$ are defined as follows:

$$A^{(j,k;\epsilon)} = \left\{ t : U^{(n_j)}(t) - U^{(n_k)}(t) \geq \epsilon \right\} \qquad \text{and} \qquad \widetilde{A}^{(j,k;\epsilon)} = \left\{ t : U^{(n_j)}(t) - U^{(n_k)}(t) \leq -\epsilon \right\}.$$

The main idea is that for each m, $U^{(m)}$ is a non-negative local martingale and consequently $\mathbb{E}\left[\langle U^{(m)} \rangle^{\alpha/2}(+\infty) \right] < C(\alpha)$ for $\alpha \in (0,1)$, where the bound does not depend on m. This therefore gives uniform restrictions on the set of crossing times that cross ϵ; the set of crossing times has Lebesgue measure 0 in the time variable.

Note that $\widetilde{A}^{(j,k;\epsilon)} = A^{(k,j;\epsilon)}$. Let $B_{jk} = A^{(j,k;\epsilon)}_{\alpha_{jk}} \setminus A^{(j,k;\epsilon)}$ or $\widetilde{A}^{(j,k;\epsilon)}_{\alpha_{jk}} \setminus \widetilde{A}^{(j,k;\epsilon)}$ where some of the notation is suppressed. Let $0 < \beta < 1$. Then, using $\int_0^\infty \frac{1}{(1+t)^2} dt = 1$, Hölder's inequality gives for any positive integer valued function $m(j,k)$ of j and k (e.g. $m(j,k) = j$ or k):

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_{B_{jk}} \frac{1}{(1+t)^2} U^{(n_{m(j,k)})}(t)dt\right)^{\beta}\right] \\
&\quad \leq \mathbb{E}\left[\sup_t U^{(n_{m(j,k)})(1+\beta)/2}(t)\right]^{2\beta/(1+\beta)} \mathbb{E}\left[\left(\int_{B_{jk}} \frac{1}{(1+t)^2} dt\right)^{\beta(1+\beta)/(1-\beta)}\right]^{(1-\beta)/(1+\beta)} \\
&\quad \leq \mathbb{E}\left[U_0^{(1+\beta)/2}\right]^{2\beta/(1+\beta)} \left(\frac{2}{1-\beta}\right)^{2\beta/(1+\beta)} \mathbb{E}\left[\left(\int_{B_{jk}} \frac{1}{(1+t)^2} dt\right)^{\beta}\right]^{(1-\beta)/(1+\beta)} \qquad (23)
\end{aligned}
$$

where the bound was obtained using Corollary 2.3. The second to third line used the following: since $\int_0^\infty \frac{1}{(1+t)^2} dt = 1$, it follows that $\int_{B_{jk}} \frac{1}{(1+t)^2} dt \leq 1$. Since $\frac{1+\beta}{1-\beta} > 1$, it follows that $\frac{1+\beta}{1-\beta}$ may be removed from the exponent $\frac{\beta(1+\beta)}{1-\beta}$ going from the second to the third line of (23).

The first task is to show that for any $\beta < 1$, the collection of random variables

$$
\Xi_{jk} := \left(\int_{B_{jk}} \frac{1}{(1+t)^2} U^{(n_{m(j,k)})}(t)dt\right)^{\beta}
$$

is uniformly integrable, where m is a positive integer valued function, of two positive integers. This is seen as follows: let $U^{(n_k)*} = \sup_t U_t^{(n_k)}$, then taking supremum over positive integer valued functions m in the definition of Ξ, and using m^* to denote the specific (non-random) positive integer valued function which gives the maximum from the second line onwards,

$$
\begin{aligned}
&\sup_{j,k,m} \mathbb{E}\left[\Xi_{jk} \mathbf{1}_{\{\Xi_{jk} \geq N\}}\right] \\
&\quad \leq \sup_{j,k} \mathbb{E}\left[U^{(n_{m^*(j,k)})*\beta} \mathbf{1}_{\{U^{(n_{m^*(j,k)})*} \geq N^{1/\beta}\}}\right] \\
&\quad \leq \sup_{j,k} \mathbb{E}\left[U^{(n_{m^*(j,k)})*(1+\beta)/2}\right]^{2\beta/(1+\beta)} \mathbb{Q}(U^{(n_{m(j,k)})*} \geq N^{1/\beta})^{(1-\beta)/(1+\beta)} \\
&\quad \leq \left(\frac{2}{1-\beta}\right)^{2\beta/(1+\beta)} \mathbb{E}\left[U_0^{(1+\beta)/2}\right]^{2\beta/(1+\beta)} \mathbb{Q}\left(U^{(n_{m^*(j,k)})*(1+\beta)/2} \geq N^{(1+\beta)/2\beta}\right)^{(1-\beta)/(1+\beta)} \\
&\quad \leq \frac{1}{N^{(1-\beta)/2\beta}} \mathbb{E}\left[U_0^{(1+\beta)/2}\right] \frac{2}{1-\beta} \overset{N \to +\infty}{\longrightarrow} 0
\end{aligned}
$$

using Corollary 2.3 directly to deal with the first term and Markov's inequality followed by Corollary 2.3 on the second. Uniform integrability has been established.

Set

$$
a_{jk}(t) = 0 \vee \frac{t(1+\alpha_{jk}) - \alpha_{jk}}{1 + (1+t)\alpha_{jk}}
$$

and

$$
b_{jk}(t) = \begin{cases} \frac{t+(1+t)\alpha_{jk}}{1-(1+t)\alpha_{jk}} & t < \frac{1}{\alpha_{jk}} - 1 \\ +\infty & t \geq \frac{1}{\alpha_{jk}} - 1. \end{cases}
$$

Then $\lim_{j\to+\infty}\left(\lim_{k\to+\infty} a_{jk}(t)\right)=t$ and $\lim_{j\to+\infty}\left(\lim_{k\to+\infty} b_{jk}(t)\right)=t$ almost surely. Let

$$U^{(j,k)} := U^{(n_j)} - U^{(n_k)} \tag{24}$$

then:

$$A^{(j,k;\epsilon)}_{\alpha_{jk}} \backslash A^{(j,k;\epsilon)} = \left\{ t : U^{(j,k)}(t) < \epsilon, \qquad \sup_{s\in[a_{jk}(t),b_{jk}(t)]} U^{(j,k)}(s) \geq \epsilon \right\}$$

and:

$$\mathbb{E}\left[\int_{A^{(j,k;\epsilon)}_{\alpha_{jk}} \backslash A^{(j,k;\epsilon)}} \frac{1}{(1+t)^2} dt\right] = \int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(\left\{U^{(j,k)}(t)<\epsilon\right\} \bigcap \left\{ \sup_{a(t)\leq s\leq b(t)} U^{(j,k)}(s) \geq \epsilon \right\}\right) dt.$$

For a process X let:

$$X^{+_{jk}}(t) = \sup_{a_{jk}(t)\leq s\leq b_{jk}(t)} X(s) \tag{25}$$

so that, with this notation,

$$\mathbb{E}\left[\int_{A^{(j,k;\epsilon)}_{\alpha_{jk}} \backslash A^{(j,k;\epsilon)}} \frac{1}{(1+t)^2} dt\right] = \int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(\left\{U^{(j,k)}(t)<\epsilon \leq U^{(j,k)+_{jk}}(t)\right\}\right) dt.$$

Note that:

- $\lim_{j\to+\infty}\left(\lim_{k\to+\infty} \alpha_{jk}\right)=0$ almost surely.
- $U^{(jk)}$ is a family of local martingales satisfying

$$\mathbb{E}\left[\langle U^{(jk)}\rangle(\infty)^{\alpha/2}\right] \leq \mathbb{E}\left[\left(2\langle U^{(n_j)}\rangle(\infty) + 2\langle U^{(n_k)}\rangle(\infty)\right)^{\alpha/2}\right] \leq 2^{1+(\alpha/2)}\widetilde{K}(\alpha)$$

for $0<\alpha<1$, where $\widetilde{K}(\alpha)$ is from (17).

Let $\mathcal{U}$ denote the class of local martingales U such that $\mathbb{E}\left[[U](\infty)^{\alpha/2}\right] \leq 2^{1+(\alpha/2)}\widetilde{K}(\alpha)$ for $0<\alpha<1$. This space is *complete* following with metric $D_\alpha(M,N) = \mathbb{E}\left[\sup_t |M(t)-N(t)|^\alpha\right]$ for $0<\alpha<1$, following Lemma 3.9. Then

$$\int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(\left\{U^{(j,k)}(t)<\epsilon \leq U^{(j,k)+_{jk}}(t)\right\}\right) dt \leq \sup_{U\in\mathcal{U}} \int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(\left\{U(t)<\epsilon \leq U^{+_{jk}}(t)\right\}\right) dt.$$

Let $\widetilde{U}^{(j,k)}\in\mathcal{U}$ satisfy

$$\sup_{U\in\mathcal{U}} \int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(\left\{U(t)<\epsilon \leq U^{+_{jk}}(t)\right\}\right) dt = \int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(\left\{\widetilde{U}^{(j,k)}(t)<\epsilon \leq \widetilde{U}^{(j,k)+_{jk}}(t)\right\}\right) dt$$

There exists a $\widetilde{U}^{(j,k)}\in\mathcal{U}$ which satisfies this property since $\mathcal{U}$ is complete, together with the bounded convergence theorem. Let $\widetilde{U}\in\mathcal{U}$ and $\widetilde{V}$ satisfy

$$\limsup_{j\to+\infty,k\to+\infty}\int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(\left\{\widetilde{U}^{(j,k)}(t)<\epsilon\le\widetilde{U}^{(j,k)+_{jk}}(t)\right\}\right)dt$$
$$=\int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(\left\{\widetilde{U}(t)<\epsilon\le\widetilde{V}(t)\right\}\right)dt.$$

The existence of the pair $(\widetilde{U},\widetilde{V})$ follows from the completeness of $\mathcal{U}$, together with the bounded convergence theorem. Consider a sequence (j_n,k_n) which gives the supremum. There exists a subsequence $(\widetilde{U}^{(j_n,k_n)},\widetilde{U}^{(j_n,k_n)+_{j_nk_n}})_{n\ge1}$ and a limit $(\widetilde{U},\widetilde{V})$; bounded convergence gives that the limit of the integrals is the integral of the limit. Let $\widetilde{U}^*(t)=\lim_{\delta\downarrow0}\sup_{s\in[t-\delta\vee0,t+\delta]}\widetilde{U}(s)$ and $\widetilde{U}_*(t)=\lim_{\delta\downarrow0}\inf_{s\in[t-\delta\vee0,t+\delta]}\widetilde{U}(s)$. Since $\lim_{j,k}a_{jk}(t)=\lim_{jk}b_{jk}(t)=t$, it follows, using the Bounded Convergence lemma to get from line 3 to line 4, that:

$$\begin{aligned}
&\limsup_{j,k\to+\infty}\int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(U^{(j,k)}(t)<\epsilon\le U^{(j,k;+_{jk})}(t)\right)dt\\
&\quad\le\int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(\widetilde{U}_*(t)<\epsilon\le\widetilde{U}^*(t)\right)dt\\
&\quad=\int_0^\infty \frac{1}{(1+t)^2}\mathbb{E}\left[\lim_{\delta\downarrow0}\mathbf{1}_{\{\widetilde{U}_*(t)<\epsilon-\delta<\epsilon\le\widetilde{U}^*(t)\}}\right]dt\\
&\quad=\lim_{\delta\downarrow0}\int_0^\infty \frac{1}{(1+t)^2}\mathbb{Q}\left(\widetilde{U}_*(t)<\epsilon-\delta<\epsilon\le\widetilde{U}^*(t)\right)dt=0.
\end{aligned}$$

It is equal to zero, because for any $\delta>0$, a local martingale of bounded quadratic variation has at most a finite number of jumps of magnitude greater than δ, hence for any $\delta>0$, the probability of having a jump of size greater than δ is only greater than 0 on a set of $\frac{dt}{(1+t)^2}$ measure 0.

For $B_{jk}=A_{\alpha_{jk}}^{(j,k;\epsilon)}\backslash A^{(j,k;\epsilon)}$, it has now been established that for all $\epsilon>0$,

$$\mathbb{E}\left[\int_{B_{jk}}\frac{1}{(1+t)^2}dt\right]\overset{j\to+\infty,k\to+\infty}{\longrightarrow}0.$$

The argument is similar for $B_{jk}=\widetilde{A}_{\alpha_{jk}}^{(j,k;\epsilon)}\backslash\widetilde{A}^{(j,k;\epsilon)}$.

It now follows from Equation (23) that for all $\beta\in(0,1)$ where m is a function of j and k (e.g. $m=j$ or k),

$$\mathbb{E}\left[\left(\int_{B_{jk}}\frac{1}{(1+t)^2}U^{(n_{m(j,k)})}(t)dt\right)^\beta\right]\overset{j,k\to+\infty}{\longrightarrow}0.$$

By virtue of uniform integrability, it now follows that, for $m(j,k)=j$ or k, $\left(\int_{B_{jk}}\frac{1}{(1+t)^2}U^{(n_{m(j,k)})}(t)dt\right)^\beta$ converges to 0 in L^1 for all $\beta\in(0,1)$.

It has now been established that the 'Prohorov jiggle' does not contribute in the sense that if $\rho(u^{(n_j)},u^{(n_k)})\to0$ almost surely, then for any $\epsilon>0$,

$$\int_0^\infty \frac{1}{(1+t)^2}|U^{(n_j)}(t)-U^{(n_k)}(t)|\mathbf{1}_{\{|U^{(n_j)}(t)-U^{(n_k)}(t)|\geq\epsilon\}}dt \to 0 \qquad \mathbb{Q}-\text{almost surely}$$

hence

$$\int_0^\infty \frac{1}{(1+t)^2}|U^{(n_j)}(t)-U^{(n_k)}(t)|dt \to 0 \qquad \mathbb{Q}-\text{almost surely}$$

Let $\mathcal{A}_{jk} = \int_0^\infty \frac{1}{(1+t)^2}|U^{(n_j)}(t)-U^{(n_k)}(t)|dt$. For $0<\alpha<1$,

$$\begin{aligned}\sup_{jk}\mathbb{E}\left[\mathcal{A}_{jk}^\alpha \mathbf{1}_{\{A_{jk}^\alpha>N\}}\right] &\leq \sup_{jk}\mathbb{E}\left[\mathcal{A}_{jk}^{(1+\alpha)/2}\right]^{2\alpha/(1+\alpha)}\mathbb{Q}\left(\mathcal{A}_{jk}^\alpha\geq N\right)^{(1-\alpha)/(1+\alpha)}\\ &\leq \left(\frac{4}{1-\alpha}\right)\mathbb{E}\left[U_0^{(1+\alpha)/2}\right]\frac{1}{N^{(1-\alpha)/2\alpha}} \stackrel{N\to+\infty}{\longrightarrow} 0\end{aligned}$$

hence uniform integrability holds from which, for all $0<\alpha<1$,

$$\lim_{j\to+\infty}\lim_{k\to+\infty}\mathbb{E}\left[\left(\int_0^\infty \frac{1}{(1+t)^2}|U^{(n_j)}(t)-U^{(n_k)}(t)|dt\right)^\alpha\right]=0. \tag{26}$$

Again, using the notation $U^{(j,k)}=U^{(n_j)}-U^{(n_k)}$, set $f^{(jk)}(t)=\langle U^{(jk)}\rangle(t)$ and $C^{(j,k)}(s)=\inf\{t: f^{(jk)}(t)\geq s\}$.

Note that for any $s<+\infty$, the stopped process $U^{(j,k)2}(t\wedge C^{(jk)}(s))-f^{(jk)}(t\wedge C^{(jk)}(s))$ is a martingale and that

$$\mathbb{E}\left[U^{(j,k)2}(t\wedge C^{(jk)}(s))\right]=\mathbb{E}\left[f^{(jk)}(t\wedge C^{(jk)}(s))\right]\leq s.$$

Furthermore, the convergence (26) gives that, almost surely, $U^{(jk)}(t)\to 0$ for Lebesgue-almost all $t\in\mathbb{R}_+$. Since for all $\alpha\in(0,1)$,

$$\mathbb{E}\left[\sup_t |U^{(j,k)}(t)|^\alpha\right]<\mathbb{E}\left[\sup_t U^{(n_j)}(t)^\alpha\right]+\mathbb{E}\left[\sup_t U^{(n_k)}(t)^\alpha\right]<2\widetilde{K}(\alpha),$$

it follows that for any $s<+\infty$,

$$\mathbb{E}\left[\int_0^{C^{(jk)}(s)}\frac{1}{(1+t)^2}U^{(j,k)2}(t)dt\right]\stackrel{j,k\to\infty}{\longrightarrow} 0.$$

It follows from Lemma 4.3 below that:

$$\mathbb{E}\left[\int_0^{C^{(jk)}(s)}\frac{1}{(1+t)^2}U^{(j,k)2}(t)dt\right]=\mathbb{E}\left[\int_0^{C^{(jk)}(s)}\frac{1}{(1+t)^2}f^{(jk)}(t)dt\right]$$

from which it follows that for all $s<+\infty$, $\sup_{t:f^{(jk)}(t)<s}f^{(jk)}(t)\stackrel{j,k\to+\infty}{\longrightarrow} 0$.

The following argument (similar to arguments given previously) shows that $f^{(i,j)\alpha/2}(+\infty)$ is uniformly integrable for all $0<\alpha<1$:

$$\sup_{i,j} \mathbb{E}\left[f^{(i,j)\alpha/2}(+\infty)\mathbf{1}_{\{f^{(i,j)\alpha/2}(+\infty)\geq N\}}\right] \leq \frac{1}{N^{(1-\alpha)/2\alpha}} \sup_{ij} \mathbb{E}\left[f^{(i,j)(1+\alpha)/2}(+\infty)\right],$$

while (using $2x^2+2y^2 \geq (x-y)^2$, and for $\alpha\in(0,1)$ and $x,y>0$ $(x+y)^\alpha < x^\alpha + y^\alpha$)

$$\begin{aligned}
\sup_{ij} \mathbb{E}\left[f^{(i,j)(1+\alpha)/2}(+\infty)\right] &\leq 2^{(3+\alpha)/2} \sup_j \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} v^{(n_j)2\gamma}(t,x)dxdt\right)^{(1+\alpha)/2}\right] \\
&\leq 2^{(3+\alpha)/2}\widetilde{K}\left(\frac{1+\alpha}{2}\right) < +\infty.
\end{aligned}$$

From this, uniform integrability follows, hence $\mathbb{Q}(f^{(ij)}(+\infty) < N) \overset{N\to+\infty}{\longrightarrow} 1$ and $f^{(ij)}(+\infty) \overset{i,j\to+\infty}{\longrightarrow} 0$ almost surely. From this, it follows from representation (21) that:

$$\lim_{i\to+\infty}\left(\lim_{j\to+\infty} \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} (\sigma_i(t,x)v^{(n_i)\gamma}(t,x) - \sigma_j(t,x)v^{(n_j)\gamma}(t,x))^2 dxdt \right)^{\alpha/2}\right]\right) = 0 \qquad (27)$$

where the functions $\sigma_j : \Omega\times\mathbb{R}_+\times\mathbb{S}^1 \to \{-1,+1\}$ adapted to $(\mathcal{G}_t)_{t\geq 0}$ are defined in (21). Since $v^{(n_i)} \geq 0$, it follows from (27) that

$$\begin{aligned}
&\lim_{i\to+\infty}\left(\lim_{j\to+\infty} \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} (v^{(n_i)\gamma}(t,x) - v^{(n_j)\gamma}(t,x))^2 dxdt \right)^{\alpha/2}\right]\right) \\
&\quad\leq \lim_{i\to+\infty}\left(\lim_{j\to+\infty} \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} (\sigma_i(t,x)v^{(n_i)\gamma}(t,x) - \sigma_j(t,x)v^{(n_j)\gamma}(t,x))^2 dxdt \right)^{\alpha/2}\right]\right) = 0,
\end{aligned}$$

from which it follows that the sequence $(v^{(n_j)\gamma})_{j\geq 0}$ is Cauchy in $d_{2,\alpha}$ for $\alpha\in(0,1)$. Since

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} (v^{(n_i)\gamma}(t,x) - v^{(n_j)\gamma}(t,x))^2 dxdt \right)^{\alpha/2}\right] \\
&\qquad\qquad \geq \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}^1} (v^{(n_i)}(t,x) - v^{(n_j)}(t,x))^{2\gamma} dxdt \right)^{\alpha/2}\right],
\end{aligned}$$

it follows that the sequence $(v^{(n_j)})_{j\geq 0}$ is Cauchy in the space $\mathcal{S}_{2\gamma,\alpha}$ and since the space is complete (by Lemma 3.8), the sequence therefore has a limit, which is u. It follows that $\lim_{j\to+\infty} d_{2,\alpha}(v^{(n_j)\gamma}, u^\gamma) = 0$ and $\lim_{j\to+\infty} d_{2\gamma,\alpha}(v^{(n_j)}, u) = 0$. Lemma 4.2 is proved. □

The argument required the following lemma:

Lemma 4.3. *Let M be a continuous local martingale, with quadratic variation process $\langle M\rangle$ and let C denote its inverse function;*

$$C(s) = \inf\{t : \langle M\rangle(t) \geq s\}.$$

Let

$$W(s) = \begin{cases} M(C(s)) & 0 \leq s \leq \langle M \rangle(+\infty) \\ M(+\infty) + B(s) - B(\langle M \rangle(\infty)) & s > \langle M \rangle(+\infty) \end{cases} \tag{28}$$

where B is a Wiener process independent of M. We assume that the probability space has been extended to accommodate B and we consider only this extended probability space. Then W is a Wiener process, which is independent of $\langle M \rangle$. In particular, for deterministic functions g and $s \in \mathbb{R}_+$,

$$\mathbb{E}\left[\int_0^{C(s)} g(r) M^2(r) dr\right] = \mathbb{E}\left[\int_0^{C(s)} g(r) \langle M \rangle(r) dr\right].$$

Proof Note: W is the Dambis, Dubins-Schwartz Wiener process, or DDS Wiener process, of M. Let

$$\mathcal{K}_t = \sigma(\{\langle M \rangle(s) : 0 \leq s \leq C(t)); \qquad \mathcal{K} = \cup_{t>0} \mathcal{K}_t$$

and

$$\mathcal{H}_t = \sigma(\{M(C(v)) - M(C(u)) : 0 \leq u \leq v \leq t\} \cup \{M(0)\})$$

Let W satisfy (28). Let $\{\mathcal{J}_s : 0 \leq s < +\infty\}$ denote the filtration generated by W. Then (Revuz and Yor [10] chapter V page 182 Theorem 1.7) W is a standard Wiener process with respect to $\{\mathcal{J}_s : 0 \leq s < +\infty\}$ where

$$\mathcal{J}_s = \begin{cases} \mathcal{K}_s \otimes \mathcal{H}_s & 0 \leq s \leq \langle M \rangle(+\infty) \\ \mathcal{K}_{\langle M \rangle(+\infty)} \otimes \mathcal{H}_{\langle M \rangle(+\infty)} \otimes \sigma(\{B(v) - B(u) : \langle M \rangle(+\infty) \leq u \leq v \leq s\}) & s > \langle M \rangle(+\infty) \end{cases}$$

Now let $\mathcal{J} = \cup_{s \geq 0} \mathcal{J}_s$ and consider the filtered probability space $(\Omega, (\mathcal{J}_s)_{s\geq 0}, \mathcal{J}, \mathbb{Q})$ where $\mathbb{Q}$ has been extended to accommodate the Wiener process B. $(\mathcal{K}, \mathbb{Q})$-almost surely, the process W clearly satisfies Protter's definition of a semimartingale (page 44 of Protter [9]), adapted to $((\mathcal{H}_s)_{s\geq 0}, \mathbb{Q}(.|\mathcal{K}))$. By the Bichteler-Dellacherie theorem (Protter [9](2004) p.114), it is a *classical semimartingale* with respect to this filtration (Protter's definition) and has decomposition

$$W(t) = A(t) + N(t) \tag{29}$$

where A is a process of bounded variation and N is a $((\mathcal{H}_s)_{s\geq 0}, \mathbb{Q}(.|\mathcal{K}))$ martingale, $\mathbb{Q}$ almost surely. Since $\langle W \rangle(t) = t$, it follows that $\langle N \rangle(t) = t$ and hence N is a $\mathbb{Q}(.|\mathcal{K})$ Wiener process, $\mathbb{Q}$ almost surely.

The next task is to show that $N = W$. To do this, consider the process

$$\widetilde{N}_s(t) = \mathbb{E}[N(s)|\mathcal{K}_t \otimes \mathcal{H}_s] \qquad t \geq s.$$

For fixed s, this is a $(\mathcal{K}_t \otimes \mathcal{H}_s)_{t\geq s}$ continuous martingale and by the Martingale Representation Theorem, it follows that $(N_s(t))_{t\geq s}$ has representation

$$\widetilde{N}_s(t) = \widetilde{N}_s(s) + \int_s^t f(s,r) d\widetilde{B}_r$$

where (fixed s with t increasing), $(\widetilde{B}_t)_{t\geq s}$ is a $(\mathcal{K}_t \otimes \mathcal{H}_s)_{t\geq s}$ adapted standard Wiener process and $(f(s,t))_{t\geq s}$ is predictable with respect to $(\mathcal{K}_t \otimes \mathcal{H}_s)_{t\geq s}$. Now, since $(\mathcal{K}_t)_{g\geq 0}$ is the filtration generated by the increasing process, it therefore does not support a Wiener process. This follows in a straightforward manner from arguments similar to those for the Martingale Representation Theorem, Lemma (3.1) and Proposition (3.2) on pages 198 and 199 from Chapter V of Revuz and Yor [10]: firstly, consider functions of the type

$$f = \sum_{j=1}^{n} \lambda_j \mathbf{1}_{(t_{j-1},t_j]}.$$

Let $\mathcal{S}$ be the set of such step functions with compact support in $\mathbb{R}_+$ and set

$$\mathcal{E}^f(T) = \exp\left\{\int_0^{C(T)} f(s)d\langle M\rangle(t)\right\}.$$

Then the set $\{\mathcal{E}^f(T) : f \in \mathcal{S}\}$ is total in $L^2(\mathcal{K}_T, \mathbb{Q})$ and hence any random variable in $L^2(\mathcal{K}_T, \mathbb{Q})$ has representation $C + \int_0^{C(T)} a(s)d\langle M\rangle(s)$ for a function a adapted to $(\mathcal{K}_s)_{s\geq 0}$. It follows that any Brownian motion $\widetilde{B}$ adapted to $(\mathcal{K}_t)_{t\geq 0}$ satisfies

$$\widetilde{B}_t - \widetilde{B}_s = \int_{C(s)}^{C(t)} a(r)d\langle M\rangle(r) \qquad 0 < s < t$$

for a function a adapted to $(\mathcal{K}_t)_{t\geq 0}$ and hence is of bounded variation, which is a contradiction.

It follows that $\widetilde{N}_s(t) = \widetilde{N}_s(s) = N(s)$. It follows that $W(t) = \widetilde{N}_t(t) + A(t)$. Since both $(W(t))_{t\geq 0}$ and $(N(t))_{t\geq 0}$ are adapted to $(\mathcal{J}_t)_{t\geq 0}$, it follows that the same is true of A. Furthermore, N and W are both Wiener process adapted to $(\mathcal{J}_t)_{t\geq 0}$ and $\langle W, N\rangle(t) = t$, from which it follows that $\langle W - N\rangle(t) \equiv 0$ for all $t \geq 0$, hence $W = N$ as required.

Let α and β be two deterministic functions. From the above, it follows that

$$\mathbb{E}\left[e^{\int_0^\infty \alpha(s)dW(s) - \int_0^\infty \beta(s)dC(s)}|\mathcal{K}\right] = e^{\frac{1}{2}\int_0^\infty \alpha^2(s)ds - \int_0^\infty \beta(s)dC(s)}.$$

It follows that the Laplace functional is of product form and hence the independence statement, $W \perp \langle M\rangle$ follows. It now follows that, for deterministic functions g:

$$\begin{aligned}
\mathbb{E}\left[\int_0^{C(s)} g(r)M^2(r)dr\right] &= \mathbb{E}\left[\int_0^s g(C(r))M^2(C(r))dC(r)\right] = \mathbb{E}\left[\int_0^s g(C(s))\mathbb{E}[W^2(r)|\mathcal{K}]dC(r)\right] \\
&= \mathbb{E}\left[\int_0^s g(C(r))rdC(r)\right] = \mathbb{E}\left[\int_0^{C(s)} g(r)\langle M\rangle(r)dr\right]
\end{aligned}$$

as required. □

Theorem 4.4. *The limiting object u provides a solution to (1).*

Proof Consider the space of test functions

$$\mathcal{T} = \left\{ \phi : C^{\infty}(\mathbb{R}_+ \times \mathbb{S}^1) \middle| \sup_{t,x} |\phi(t,x)| + \sup_{t,x} |\phi_t(t,x) + \sup_{t,x} |\phi_{xx}(t,x)| \leq 1 \right\}$$

where ϕ_t denotes the derivative of ϕ with respect to t and ϕ_{xx} denotes the second derivative of ϕ with respect to x. A function $u^{(n)}$ satisfies Equation (16) if and only if for all $\phi \in \mathcal{T}$,

$$\begin{aligned}
&\int_{\mathbb{S}^1} u^{(n)}(t,x)\phi(t,x)dx - \int_0^t \int_{\mathbb{S}^1} u^{(n)}(s,x)\phi_s(s,x)dxds \\
&\quad -\frac{1}{2}\int_0^t \int_{\mathbb{S}^1} \phi_{xx}(s,x)u^{(n)}(s,x)dxds = \int_{\mathbb{S}^1} u_0(x)\phi(0,x)dx + \int_0^t \int_{\mathbb{S}^1} \phi(s,x)v^{(n)\gamma}(s,x)W(dx,ds)
\end{aligned}$$

where (as usual) $v^{(n)} = u^{(n)} \wedge n$. A function u satisfies Equation (15) if and only if for all $\phi \in \mathcal{T}$

$$\begin{aligned}
&\int_{\mathbb{S}^1} u(t,x)\phi(t,x)dx - \int_0^t \int_{\mathbb{S}^1} u(s,x)\phi_s(s,x)dxds \\
&\quad -\frac{1}{2}\int_0^t \int_{\mathbb{S}^1} \phi_{xx}(s,x)u(s,x)dxds = \int_{\mathbb{S}^1} u_0(x)\phi(0,x)dx + \int_0^t \int_{\mathbb{S}^1} \phi(s,x)u^{\gamma}(s,x)W(dx,ds).
\end{aligned}$$

From the foregoing, it is clear that

$$\int_0^{\infty} \int_{\mathbb{S}^1} \phi_s(s,x)u^{(n_j)}(s,x)dxds \stackrel{j\rightarrow+\infty}{\longrightarrow} \int_0^{\infty} \int_{\mathbb{S}^1} \phi_s(s,x)u(s,x)dxds$$

and

$$\int_0^{\infty} \int_{\mathbb{S}^1} \phi_{xx}(s,x)u^{(n_j)}(s,x)dsdx \stackrel{j\rightarrow+\infty}{\longrightarrow} \int_0^{\infty} \int_{\mathbb{S}^1} \phi_{xx}(s,x)u(s,x)dsdx.$$

For the last term,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t<+\infty}\left|\int_0^t \int_{\mathbb{S}^1} \phi(s,x)v^{(n_j)\gamma}(s,x)W(ds,dx) - \int_0^t \int_{\mathbb{S}^1} \phi(s,x)u^{\gamma}(s,x)W(ds,dx)\right|^{\alpha}\right] \\
&\quad \leq C(\alpha)\mathbb{E}\left[\left(\int_0^{\infty} \int_{\mathbb{S}^1} \phi^2(s,x)\left(v^{(n_j)\gamma}(s,x) - u^{\gamma}(s,x)\right)^2 dsdx\right)^{\alpha/2}\right] \\
&\quad \stackrel{j\rightarrow+\infty}{\longrightarrow} 0
\end{aligned}$$

by Lemma 4.2 and the definition of the stochastic integral. The result follows. ☐

5 Uniqueness

This section is devoted to the proof of the following uniqueness result:

Theorem 5.1 (Uniqueness). *Let u and v denote two solutions to Equation (1) in $\mathcal{S}_{2\gamma,\alpha}$ for all $\alpha \in (0,1)$. Suppose that $u(0,.) = v(0,.)$ and suppose that there is a positive constant C such that $\max_{x\in\mathbb{S}^1} u(0,x) < C < +\infty$. Then $d_{2\gamma,\alpha}(u,v) \equiv 0$ for all $\alpha \in (0,1)$.*

Proof Any function $u \in \mathcal{S}_{2\gamma,\alpha}$ for some $\alpha \in (0,1)$ satisfies: $\int_0^\infty \|u(t,.)\|_{2\gamma}^{2\gamma} dt < +\infty$ $\mathbb{Q}$-almost surely. On the set of $\mathbb{Q}$-measure 1 where $\int_0^\infty \|u(t,.)\|_{2\gamma}^{2\gamma} dt < +\infty$, clearly $\|u(t,.)\|_{2\gamma} < +\infty$ for Lebesgue - almost all $t \in \mathbb{R}_+$ and hence $\|u(t,.)\|_2 < +\infty$ on this set. Denote by $\mathcal{D} \subseteq \Omega \times \mathbb{R}_+$ the set:

$$\mathcal{D} = \cup_{K>0} \left\{ \int_0^\infty \|u(t,.)\|_{2\gamma}^{2\gamma} dt + \int_0^\infty \|v(t,.)\|_{2\gamma}^{2\gamma} dt < K \right\},$$

then $\mathbb{Q}(\mathcal{D}) = 1$. For $\omega \in \mathcal{D}$, let

$$\mathcal{T}(\omega) = \cup_{K>0} \{t : \|u(t,.)\|_{2\gamma} + \|v(t,.)\|_{2\gamma} < K\}$$

then $\mathbb{Q}$-a.s., $\mathcal{T}(\omega)$ is a set of full Lebesgue measure. Let

$$\Xi = \{(\omega, t) : \omega \in \mathcal{D}, t \in \mathcal{T}(\omega)\}.$$

For $\gamma > 1$ (the situation under consideration here) for $f : \mathbb{S}^1 \to \mathbb{R}_+$, $\|f\|_2 \leq \|f\|_{2\gamma}$ by Hölder's inequality.

Let

$$\lambda_j(t) = \int_{\mathbb{S}^1} e^{-i2\pi jx} u(t,x) dx, \qquad \mu_j(t) = \int_{\mathbb{S}^1} e^{-i2\pi jx} v(t,x) dx \qquad \forall j \in \mathbb{Z}.$$

Since $\sup_t \int_{\mathbb{S}^1} u(t,x) dx < +\infty$ $\mathbb{Q}$-a.s. and $\sup_t \int_{\mathbb{S}^1} v(t,x) dx < +\infty$ $\mathbb{Q}$-a.s., it follows directly that $\max_j \sup_t (|\lambda_j(t)| + |\mu_j(t)|) < +\infty$ $\mathbb{Q}$-a.s..

Let $\widehat{u}_N(t,x) = \sum_{j=-N}^N \lambda_j(t) e^{ij2\pi x}$ and $\widehat{v}_N(t,x) = \sum_{j=-N}^N \mu_j(t) e^{ij2\pi x}$. On $\mathcal{D}$, let

$$\widehat{u} = \begin{cases} \lim_{N\to+\infty} \widehat{u}_N & \text{limit well defined} \\ 0 & \text{otherwise} \end{cases}, \widehat{v} = \begin{cases} \lim_{N\to+\infty} \widehat{v}_N & \text{limit well defined} \\ 0 & \text{otherwise} \end{cases}.$$

and let $\widehat{u} \equiv 0$ and $\widehat{v} \equiv 0$ on $\Omega \backslash \mathcal{D}$.

Justification of the Fourier Transform Firstly, by Carlesson's theorem [1](1966), the Fourier expansion of any L^2 function converges almost everywhere, hence for $(\omega, t) \in \Xi$, $\widehat{u}_N$ and $\widehat{v}_N$ converge to u and v respectively for almost all $x \in \mathbb{S}$. Secondly, norm convergence of $\widehat{u}_N$ and $\widehat{v}_N$ is standard on Ξ in the sense that

$$\lim_{N\to+\infty} \int_{\mathbb{S}} |u - \widehat{u}_N|^2(t,x) dx = \lim_{N\to+\infty} \int_{\mathbb{S}} |v - \widehat{v}_N|^2(t,x) dx = 0 \qquad (\omega, t) \in \Xi$$

and hence

$$\|u(t,.) - \widehat{u}(t,.)\|_2 = 0 \qquad \text{and} \qquad \|v(t,.) - \widehat{v}(t,.)\|_2 = 0 \qquad \forall (\omega, t) \in \Xi.$$

since for $(\omega, t) \in \mathcal{D}$, $u, v \in L^2(\mathbb{S}^1)$. This is the Riesz-Fisher theorem.

Since $u(t,x) - \widehat{u}(t,x) = 0$ $\mathbb{Q} \times dt \times dx$ almost everywhere, $\widehat{u} = \widehat{v} = 0$ on the set where it does not converge, $\mathbb{E}\left[\left(\int_0^\infty \|u\|_{2\gamma}^{2\gamma}(t) dt\right)^\alpha\right] < +\infty$ and $\mathbb{E}\left[\left(\int_0^\infty \|v\|_{2\gamma}^{2\gamma}(t) dt\right)^\alpha\right] < +\infty$ for $\alpha \in (0,1)$, it follows that for all $\alpha \in (0,1)$,

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}} |u(t,x)-\widehat{u}(t,x)|^{2\gamma} dxdt\right)^{\alpha/2}\right]=0, \qquad \mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{S}} |v(t,x)-\widehat{v}(t,x)|^{2\gamma} dxdt\right)^{\alpha/2}\right]=0$$

and hence

$$d_{2\gamma,\alpha}(v,u)=d_{2\gamma,\alpha}(\widehat{v},\widehat{u}) \qquad \forall \alpha\in(0,1).$$

Note that:

$$\|\widehat{u}(t,.)\|_2^2=\sum_{-\infty}^{\infty}\lambda_j(t)\lambda_{-j}(t), \qquad \|\widehat{v}(t,.)\|_2^2=\sum_{-\infty}^{\infty}\mu_j(t)\mu_{-j}(t) \qquad \forall(\omega,t)\in\Xi.$$

An Infinite Dimensional Itô Formula Let $\lambda_{j0}:=\lambda_j(0)$ and $\mu_{j0}:=\mu_j(0)$ so that $u(0,x)=\sum_{j=-\infty}^{\infty}\lambda_{j0}e^{ij2\pi x}$ and $v(0,x)=\sum_{j=-\infty}^{\infty}\mu_{j0}e^{ij2\pi x}$. By integration over the space variable and using that both u and v satisfy the equation $w_t=\frac{1}{2}w_{xx}+w^\gamma\xi$,

$$\begin{cases} \lambda_n(t)=\lambda_{n0}-\frac{n^2}{2}\int_0^t\lambda_n(s)ds+M_n(t)\\ \mu_n(t)=\lambda_{n0}-\frac{n^2}{2}\int_0^t\mu_n(s)ds+N_n(t)\end{cases} \tag{30}$$

where $M_n(t)=\int_0^t\int_{\mathbb{S}^1}e^{-inx}u^\gamma(s,x)W(dx,ds)$ and $N_n(t)=\int_0^t\int_{\mathbb{S}^1}e^{-inx}v^\gamma(s,x)W(dx,ds)$. Note that

$$\begin{cases} \langle M_m,M_n\rangle(t)=\int_0^t\int_{\mathbb{S}^1}e^{-i(n+m)x}v^{2\gamma}(s,x)dxds\\ \langle N_m,N_n\rangle(t)=\int_0^t\int_{\mathbb{S}^1}e^{-i(n+m)x}u^{2\gamma}(s,x)dxds,\\ \langle M_m,N_n\rangle(t)=\int_0^t\int_{\mathbb{S}^1}e^{-i(n+m)x}u^{\gamma}(s,x)v^\gamma(s,x)dxds.\end{cases} \tag{31}$$

The next step is establish that an Itô formula holds for functions $\mathcal{U}(\lambda,\mu)$ belonging to a suitable class. The class on which the Itô formula is established is a subset of $C_0(\mathcal{S}^2,d_{\mathcal{S}^2})$ (Definition 5.2 given later); it is functions in $C_0(\mathcal{S}^2,d_{\mathcal{S}^2})$ which also satisfy hypotheses (38) and (40). The second of these implies that the function can be approximated by restriction to finite dimensions. Denote by $\mathcal{S}$ the space:

$$\mathcal{S}=\{\gamma:\gamma_n=\gamma_{-n}^* \qquad \forall n\in\mathbb{Z}\}$$

with metric $d_{\mathcal{S}}(\gamma,\delta)=\sqrt{\sum_{n=-\infty}^{\infty}(\gamma_n-\delta_n)(\gamma_{-n}-\delta_{-n})}$. For $(\gamma_1,\gamma_2)\in\mathcal{S}^2$ and $(\delta_1,\delta_2)\in\mathcal{S}^2$, we use the notation

$$d_{\mathcal{S}^2}((\gamma_1,\gamma_2),(\delta_1,\delta_2))=\sqrt{d_{\mathcal{S}}(\gamma_1,\delta_1)^2+d_{\mathcal{S}}(\gamma_2,\delta_2)^2}.$$

The aim is to show that the stochastic evolution defined by (30) defines a Feller transition semigroup over $C_0(\mathcal{S}^2,d_{\mathcal{S}^2})$, which is now defined.

Definition 5.2. *The space $C_0(\mathcal{S}^2,d_{\mathcal{S}^2})$ is defined as the space of functions over $\mathcal{S}^2$ which satisfy the following two conditions:*

1. *they are continuous under metric $d_{\mathcal{S}}$. That is, for any sequence $(\gamma_n,\delta_n)_{n\geq1}$,*

$$d_{\mathcal{S}^2}((\gamma_n,\delta_n),(\gamma,\delta))\overset{n\to+\infty}{\longrightarrow}0\Rightarrow\lim_{n\to+\infty}|\mathcal{U}(\gamma_n,\delta_n)-\mathcal{U}(\gamma,\delta)|=0.$$

2. $\mathcal{U}(\gamma,\delta) \overset{|\gamma|+|\delta|\to+\infty}{\longrightarrow} 0.$

We now define a suitable metric on $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$. Let

$$\mathcal{U}^{(N)}(\lambda,\mu) = \mathcal{U}(\widetilde{\lambda}^{(N)}, \widetilde{\mu}^{(N)}) \tag{32}$$

where

$$\widetilde{\lambda}_j^{(N)} = \begin{cases} \lambda_j & j \in \{-N,\dots,N\} \\ 0 & \text{other} \end{cases} \qquad \widetilde{\mu}_j^{(N)} = \begin{cases} \mu_j & j \in \{-N,\dots,N\} \\ 0 & \text{other} \end{cases} \tag{33}$$

Consider the following inner product:

$$\langle \mathcal{U},\mathcal{V}\rangle = \frac{1}{e-1}\sum_{N=1}^{\infty} e^{-N}\langle \mathcal{U}^{(N)},\mathcal{V}^{(N)}\rangle_N \tag{34}$$

where $\mathcal{U}^{(N)}$ and $\mathcal{V}^{(N)}$ are the N-approximations defined by (32) for $\mathcal{U}$ and $\mathcal{V}$ and, for $f,g:\mathbb{R}^{4N+2}\to\mathbb{R}$, $\langle f,g\rangle_N$ is defined as:

$$\langle f,g\rangle_N = \int \frac{1}{(2\pi)^{2N+1}} e^{-|x|^2/2} f^{(N)}(x) g^{(N)}(x)dx \tag{35}$$

Here $x=(x_1,\dots,x_{4N+2})\in\mathbb{R}^{4N+2}$ and the components of the vector x are the $4N+2$ real valued variables required to define $\lambda_{-N},\dots,\lambda_N$ and $\mu_{-N},\dots,\mu_N$, using $\lambda_j=\lambda^*_{-j}$, $\lambda_0=x_1$, $\lambda_j = x_{2j}+ix_{2j+1}$ for $j=1,\dots,N$, $\mu_0=x_{2N+2}$, $\mu_j = x_{2j+2N+1}+ix_{2j+2N+2}$ for $j=1,\dots,N$.

The inner product defined by (34) is clearly an inner product; it satisfies

- symmetry $\langle\mathcal{U},\mathcal{V}\rangle=\langle\mathcal{V},\mathcal{U}\rangle$, for a scalar a it satisfies $\langle a\mathcal{U},\mathcal{V}\rangle = a\langle\mathcal{U},\mathcal{V}\rangle$ and for $\mathcal{U},\mathcal{V},\mathcal{W}$, $\langle\mathcal{U},\mathcal{V}+\mathcal{W}\rangle = \langle\mathcal{U},\mathcal{V}\rangle+\langle\mathcal{U},\mathcal{W}\rangle$,
- positive definiteness; $\langle\mathcal{U},\mathcal{U}\rangle\geq 0$, with equality if and only if $\mathcal{U}\equiv 0$.

Therefore, $\langle .,.\rangle$ defines an inner product over a Hilbert space $\mathcal{H}$ such that $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})\subseteq\mathcal{H}$.

We use the following metric on $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$:

$$D(\mathcal{U},\mathcal{V}) = \sqrt{\langle\mathcal{U}-\mathcal{V},\mathcal{U}-\mathcal{V}\rangle} \tag{36}$$

and the norm:

$$\|\mathcal{U}\| = \sqrt{\langle\mathcal{U},\mathcal{U}\rangle}. \tag{37}$$

The Itô formula is established on functions of $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ (Definition 5.2) which also satisfy (38) and (40):

$$\begin{cases} \sup_{\lambda,\mu\in\mathcal{S}} |\mathcal{U}(\lambda,\mu)| < \infty \\ \sup_{\lambda,\mu\in\mathcal{S}} \sum_n n^2|\lambda_n|\,|\partial_{\lambda_n}\mathcal{U}| < +\infty, \quad \sup_{\lambda,\mu\in\mathcal{S}} \sum_n n^2|\mu_n|\,|\partial_{\mu_n}\mathcal{U}| < +\infty, \\ \sup_{\lambda,\mu\in\mathcal{S}} \sum_{mn} \left(\left|\partial^2_{\mu_m\mu_n}\mathcal{U}\right| + \left|\partial^2_{\lambda_m\mu_n}\mathcal{U}\right| + \left|\partial^2_{\lambda_m\lambda_n}\mathcal{U}\right|\right) < +\infty \\ \sup_{\lambda,\mu\in\mathcal{S}} \sum_n \left(|\partial_{\mu_n}\mathcal{U}|^2 + |\partial_{\lambda_n}\mathcal{U}|^2\right) < +\infty \end{cases} \tag{38}$$

where $\partial^p_{a_1\dots a_p}$ denotes the p^{th} partial derivative with respect to the arguments $a_1,\dots,a_p$.

Let

$$\mathcal{W}_N = \mathcal{U} - \mathcal{U}^{(N)}. \tag{39}$$

where $\mathcal{U}^{(N)}$ is defined by (32). The condition that ensures $\mathcal{U}$ can be approximated by $\mathcal{U}^{(N)}$ is:

$$\begin{cases} \lim_{N\to+\infty}\sup_{\lambda,\mu\in\mathcal{S}}|\mathcal{W}_N(\lambda,\mu)| = 0 \\ \sup_{\lambda,\mu\in\mathcal{S}}\sum_n n^2|\lambda_n|\,|\partial_{\lambda_n}\mathcal{W}_N| \overset{N\to+\infty}{\longrightarrow} 0, \qquad \sup_{\lambda,\mu\in\mathcal{S}}\sum_n n^2|\mu_n|\,|\partial_{\mu_n}\mathcal{W}_N| \overset{N\to+\infty}{\longrightarrow} 0 \\ \sup_{\lambda,\mu\in\mathcal{S}}\sum_{mn}\left(|\partial^2_{\mu_m\mu_n}\mathcal{W}_N| + \left|\partial^2_{\mu_m\lambda_n}\mathcal{W}_N\right| + |\partial^2_{\lambda_m\lambda_n}\mathcal{W}_N|\right) \overset{N\to+\infty}{\longrightarrow} 0 \end{cases} \tag{40}$$

For collections $(\lambda_j)_{j=-\infty}^{\infty}$ and $(\mu_j)_{j=-\infty}^{\infty}$ such that $\lambda_j = \lambda^*_{-j}$ (complex conjugate) and $\mu_j = \mu^*_{-j}$, set

$$f(\lambda;x) := \sum_j \lambda_j e^{i2\pi jx}$$

and consider μ and λ such that

$$\begin{cases} f(\lambda,x)\geq 0 \quad \forall x\in\mathbb{S}^1, & f(\mu,x)\geq 0 \quad \forall x\in\mathbb{S}^1, \\ \int_{\mathbb{S}^1} f(\lambda,x)^{2\gamma}dx < +\infty, & \int_{\mathbb{S}^1} f(\mu,x)^{2\gamma}dx < +\infty \end{cases}$$

For such λ and μ, set

$$F_m(\lambda,\mu) = \int_{\mathbb{S}^1} e^{-i2\pi mx} f(\lambda,x)^\gamma f(\mu,x)^\gamma dx. \tag{41}$$

Note that

$$\begin{cases} \frac{d}{dt}\langle M_m, M_n\rangle(t) = F_{m+n}(\lambda(t),\lambda(t)), \quad \frac{d}{dt}\langle N_m, N_n\rangle(t) = F_{m+n}(\mu(t),\mu(t)), \\ \frac{d}{dt}\langle M_m, N_n\rangle(t) = F_{m+n}(\lambda(t),\mu(t)). \end{cases}$$

Let $\mathcal{L}$ be defined as:

$$\begin{cases} \mathcal{L}(\lambda,\mu) = -\frac{1}{2}\sum_{n=-\infty}^{\infty} n^2(\lambda_n\partial_{\lambda_n} + \mu_n\partial_{\mu_n}) \\ \qquad\qquad +\frac{1}{2}\sum_{m,n=-\infty}^{\infty}\left(F_{m+n}(\lambda,\lambda)\partial^2_{\lambda_m\lambda_n} + F_{m+n}(\mu,\mu)\partial^2_{\mu_m\mu_n} + 2F_{m+n}(\lambda,\mu)\partial^2_{\lambda_m\mu_n}\right) \end{cases} \tag{42}$$

where $\partial^p_{a_1\dots.a_p}\mathcal{U}(\lambda,\mu)$ means the p^{th} partial derivative of $\mathcal{U}$ with respect to the arguments labelled $a_1,\dots,a_p$.

Definition 5.3 (Domain of Infinitesimal Generator)**.** *Let $\mathcal{D}_*(\mathcal{L})$ be: functions $\mathcal{U}\in C_0(\mathcal{S}^2)$ which satisfy both (38) and (40). We define $\mathcal{D}(\mathcal{L})$ as: functions $\mathcal{U}$ such that for any sequence $(\mathcal{U}_n)$ such that $D(\mathcal{U},\mathcal{U}_n)\overset{n\to+\infty}{\longrightarrow} 0$, where $\mathcal{U}_n\in\mathcal{D}_*(\mathcal{L})$, $\mathcal{L}\mathcal{U}_n\to\mathcal{Y}$ for some $\mathcal{Y}$ and we define $\mathcal{L}\mathcal{U}=\mathcal{Y}$. The space $\mathcal{D}(\mathcal{L})$ is the* domain of the infinitesimal generator $\mathcal{L}$. *The space $\mathcal{D}(\mathcal{L})$ is: functions $\mathcal{U}\in C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ on which $\mathcal{L}\mathcal{U}$ is well defined and bounded. From the definition, $\mathcal{L}\mathcal{U}$ is well defined for all $\mathcal{U}\in\mathcal{D}_*(\mathcal{L})$.*

Lemma 5.4. *$\mathcal{D}_*(\mathcal{L})$ is dense in $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ under the metric D defined by (36).*

Proof Clear. Functions in $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ are bounded and the construction of the metric ensures the convergence. A function $\mathcal{U} \in C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ may be approximated by the approximations $\mathcal{U}^{(N)}$ defined by (32), which may be further approximated by a smoothed version, with the smoothing decreasing as $N \to +\infty$. $\square$

Lemma 5.5. *Let $\mathcal{U} \in \mathcal{D}_*(\mathcal{L})$ and let $(\lambda(t), \mu(t))$ satisfy (30) with initial conditions $\lambda(0) = \lambda$, $\mu(0) = \mu$, then Itô's formula may be applied to give:*

$$\begin{array}{l} \mathcal{U}(\lambda(t),\mu(t)) - \mathcal{U}(\lambda,\mu) - \int_0^t (\mathcal{L}\mathcal{U})(\lambda(s),\mu(s))ds \\ \quad = \sum_n \int_0^t (\partial_{\lambda_n}\mathcal{U})(\lambda(s),\mu(s))dM_n(s) + \sum_n \int_0^t (\partial_{\mu_n}\mathcal{U})(\lambda(s),\mu(s))dN_n(s) \end{array} \tag{43}$$

where, by

$$\sum_n \int_0^t (\partial_{\lambda_n}\mathcal{U})(\lambda(s),\mu(s))dM_n(s) + \sum_n \int_0^t (\partial_{\mu_n})\mathcal{U}(\lambda(s),\mu(s))dN_n(s)$$

is meant a martingale with quadratic variation process Q where

$$\begin{aligned} Q(t) = & \sum_{n_1,n_2} \int_0^t \left(\partial_{\lambda_{n_1}}\mathcal{U}\right)(\lambda(s),\mu(s)) \left(\partial_{\lambda_{n_2}}\mathcal{U}\right)(\lambda(s),\mu(s)) F_{n_1+n_2}(\lambda(s),\lambda(s))ds \\ & + \sum_{n_1,n_2} \int_0^t \left(\partial_{\mu_{n_1}}\mathcal{U}\right)(\lambda(s),\mu(s)) \left(\partial_{\mu_{n_2}}\mathcal{U}\right)(\lambda(s),\mu(s)) F_{n_1+n_2}(\mu(s),\mu(s))ds \\ & + 2\sum_{n_1,n_2} \int_0^t \left(\partial_{\lambda_{n_1}}\mathcal{U}\right)(\lambda(s),\mu(s)) \left(\partial_{\mu_{n_2}}\mathcal{U}\right)(\lambda(s),\mu(s)) F_{n_1+n_2}(\lambda(s),\mu(s))ds. \end{aligned} \tag{44}$$

Proof of Lemma 5.5 Following the line of proof taken by Revuz and Yor [10] Theorem 3.3 page 141, if $\mathcal{U}$ satisfies (38) and (40), Itô's formula may be applied to $\mathcal{U}^{(N)}(\lambda, \mu)$ defined by (32) for each $N < +\infty$. Let $\mathcal{V}(t)$ denote the right hand side of (43). Then $\mathcal{V} - \mathcal{U}^{(N)}$ is given by the right hand side of (43) with each appearance of $\mathcal{U}$ replaced by $\mathcal{W}_N$ from (39), where the local martingale term is a local martingale with quadratic variation $Q^{(N)}$, given by (44), where each appearance of $\mathcal{U}$ is replaced by $\mathcal{W}_N$.

Now note that for all m,

$$|F_m(\lambda(t),\lambda(t))| \leq \|u\|_{2\gamma}^{2\gamma}(t), \qquad |F_m(\mu(t),\mu(t))| \leq \|v\|_{2\gamma}^{2\gamma}(t), \quad |F_m(\lambda(t),\mu(t))| \leq \|u\|_{2\gamma}^{\gamma}(t)\|v\|_{2\gamma}^{\gamma}(t).$$

Recall that

$$\mathbb{E}\left[\left(\int_0^\infty \|u\|_{2\gamma}^{2\gamma}(t)dt\right)^{\alpha/2}\right] < +\infty, \qquad \mathbb{E}\left[\left(\int_0^\infty \|v\|_{2\gamma}^{2\gamma}(t)dt\right)^{\alpha/2}\right] < +\infty \qquad \forall \alpha \in (0,1).$$

Using the bounds of (38) and (40), it is therefore straightforward to apply the dominated convergence theorem to show that the bounded variation terms of $\sup_{0 \leq t \leq T} |\mathcal{V}(t) - \mathcal{U}^{(N)}(t)|$ converge to 0 almost surely for all $T < +\infty$ and the quadratic variation $Q^{(N)}(+\infty)$ of the local martingale term converges to 0 as $N \to +\infty$. From this, it follows that $\lim_{N \to +\infty} \sup_{0 < t < +\infty} |\mathcal{M}^{(N)}(t)| = 0$ where $\mathcal{M}^{(N)}$ is the local martingale part. The fact that this local martingale is a *martingale* follows from the fact that the left hand side of (43) is bounded, with bound growing linearly in t, by definition ($\mathcal{U}$ is bounded because $\mathcal{U} \in C_0(\mathcal{S}^2)$; $\mathcal{L}\mathcal{U}$ bounded from the definition of $\mathcal{D}_*(\mathcal{L})$). $\square$

Establishing the Markov Property The next step is to establish that $(\lambda(t), \mu(t))_{t\geq 0}$ is a time homogeneous Markov process with infinitesimal generator $\mathcal{L}$.

Lemma 5.6. *$\mathcal{L}$ is the infinitesimal generator of a unique Feller transition semigroup $(\mathcal{Q}_t)_{t\geq 0}$ on $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$.*

Proof $\mathcal{D}_*(\mathcal{L})$ is dense in $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$, by Lemma 5.4, in the sense described in that lemma. Furthermore, $\mathcal{D}_*(\mathcal{L}) \subseteq \mathcal{D}(\mathcal{L})$ (by definition of $\mathcal{D}(\mathcal{L})$), hence $\mathcal{D}(\mathcal{L})$ is dense in $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$. The operator $\mathcal{L}$ is a *closed* operator, which almost follows from the definition of $\mathcal{D}(\mathcal{L})$; we use the characterisation that a linear operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathcal{H}$ is a closed if and only if the domain $\mathcal{D}(\mathcal{L})$ endowed with the norm $\|\mathcal{U}\| + \|\mathcal{L}\mathcal{U}\|$ is a Banach space, i.e. a linear, normed, complete space and this is clear.

Now suppose there exists a family $\mathcal{Q}$ of transition semigroups with $\mathcal{L}$ as infinitesimal generator. Note that for any $Q \in \mathcal{Q}$,

$$\frac{Q_h - I}{h} \stackrel{h\to 0}{\longrightarrow} \mathcal{L}.$$

Furthermore, if $f \in \mathcal{D}(\mathcal{L})$, then for any $Q \in \mathcal{Q}$ and all $t > 0$,

$$\mathcal{L}Q_t f = Q_t \mathcal{L} f.$$

Suppose that $\mathcal{Q}$ has more than one element; consider two of them, $Q^{(1)}$ and $Q^{(2)}$. Let $f \in \mathcal{D}(\mathcal{L})$ and let $w(s) = Q_s^{(1)} Q_{t-s}^{(2)} f$. Then

$$\frac{d}{ds} w(s) = Q_s^{(1)} \mathcal{L} Q_{t-s}^{(2)} f - Q_s^{(1)} \mathcal{L} Q_{t-s}^{(2)} f = 0$$

giving w constant on $[0, t]$, hence (taking $s = 0$ and t), $Q^{(1)}(t) f = Q_t^{(2)} f$ for all $t > 0$. It follows that there is at most one $Q \in \mathcal{Q}$.

Since $\mathcal{L}$ is a closed operator, existence now follows from the Hille-Yosida theorem: the space $\mathcal{D}_*(\mathcal{L})$ is dense in $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$. The other condition of the Hille-Yosida theorem to be satisfied is $\|(\lambda I - \mathcal{L})^{-1}\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$. Let

$$\mathcal{L}^{(N)}(\lambda, \mu) := \mathcal{L}(\widetilde{\lambda}^{(N)}, \widetilde{\mu}^{(N)})$$

where $(\widetilde{\lambda}^{(N)}, \widetilde{\mu}^{(N)})$ are defined in (33). Then every real $\lambda > 0$ belongs to the resolvent set of $\mathcal{L}^{(N)}$ and satisfies $\|(\lambda I - \mathcal{L}^{(N)})^{-1}\|_{\mathcal{O};N} \leq \frac{1}{\lambda}$, where $\|.\|_{\mathcal{O};N}$ denotes the operator norm for $(\lambda I - \mathcal{L}^{(N)})^{-1}$. The result therefore holds in the limit. In this case, the operator norm used is:

$$\|T\|_{\mathcal{O};N} = \sup_{f \in C_0(\mathcal{S}^2, d_{\mathcal{S}^2})} \frac{\sqrt{\frac{1}{e-1} \sum_{M=1}^{N} e^{-M} \langle Tf, Tf \rangle_M}}{\sqrt{\frac{1}{e-1} \sum_{M=1}^{N} e^{-M} \langle f, f \rangle_M}}$$

where $\langle ., . \rangle_M$ is defined by (35). It is now established that the infinitesimal generator $\mathcal{L}$ generates a unique Feller transition semigroup on $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$. $\square$

Establishing that the solution to the Kolmogorov equation is identically zero for $u_0 = v_0$ Now consider the co-ordinate change $\alpha_j = \frac{1}{\sqrt{2}}(\lambda_j + \mu_j)$, $\beta_j = \frac{1}{\sqrt{2}}(\lambda_j - \mu_j)$ and set $\widetilde{\mathcal{F}}(t;\alpha,\beta) = \mathcal{F}(t;\mu,\lambda)$. Then the equation may be reformulated as:

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{\mathcal{F}}(t;\alpha,\beta) = -\frac{1}{2}\sum_{j=-\infty}^{\infty} j^2 \left(\alpha_j \frac{\partial}{\partial \alpha_j} + \beta_j \frac{\partial}{\partial \beta_j}\right)\widetilde{\mathcal{F}}(t;\alpha,\beta) \\ \quad +\frac{1}{2}\sum_{jk}\left(F_{j+k}(\frac{\alpha-\beta}{\sqrt{2}},\frac{\alpha+\beta}{\sqrt{2}}) + F_{j+k}(\frac{\alpha-\beta}{\sqrt{2}},\frac{\alpha-\beta}{\sqrt{2}}) + 2F_{j+k}(\frac{\alpha-\beta}{\sqrt{2}},\frac{\alpha+\beta}{\sqrt{2}})\right)\frac{\partial^2}{\partial\alpha_j\partial\alpha_k}\widetilde{\mathcal{F}}(t;\alpha,\beta) \\ \quad +\frac{1}{2}\sum_{jk}\left(F_{j+k}(\frac{\alpha+\beta}{\sqrt{2}},\frac{\alpha+\beta}{\sqrt{2}}) + F_{j+k}(\frac{\alpha-\beta}{\sqrt{2}},\frac{\alpha-\beta}{\sqrt{2}}) - 2F_{j+k}(\frac{\alpha+\beta}{\sqrt{2}},\frac{\alpha-\beta}{\sqrt{2}})\right)\frac{\partial^2}{\partial\beta_j\partial\beta_k}\widetilde{\mathcal{F}}(t;\alpha,\beta) \\ \quad +\sum_{jk}\left(F_{j+k}(\frac{\alpha+\beta}{\sqrt{2}},\frac{\alpha+\beta}{\sqrt{2}}) - F_{j+k}(\frac{\alpha-\beta}{\sqrt{2}},\frac{\alpha-\beta}{\sqrt{2}})\right)\frac{\partial^2}{\partial\alpha_j\partial\beta_k}\widetilde{\mathcal{F}}(t;\alpha,\beta) \\ \widetilde{\mathcal{F}}(0;\alpha,\beta) = \mathcal{U}\left(\frac{1}{\sqrt{2}}(\alpha+\beta), \frac{1}{\sqrt{2}}(\alpha-\beta)\right). \end{cases} \tag{45}$$

For $\lambda = \mu$, $\beta = 0$. It follows from (45) that $\widetilde{\mathcal{F}}(t;\alpha,0)$ satisfies:

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{\mathcal{F}}(t;\alpha,0) = \sum_{j=-\infty}^{\infty}\alpha_j\frac{\partial}{\partial\alpha_j}\widetilde{\mathcal{F}}(t;\alpha,0) + 2\sum_{jk}F_{j+k}(\frac{\alpha}{\sqrt{2}},\frac{\alpha}{\sqrt{2}})\frac{\partial^2}{\partial\alpha_j\partial\alpha_k}\widetilde{\mathcal{F}}(t;\alpha,0) \\ \widetilde{\mathcal{F}}(0;\alpha,0) = \mathcal{U}\left(\frac{\alpha}{\sqrt{2}},\frac{\alpha}{\sqrt{2}}\right) \end{cases} \tag{46}$$

Let $\mathcal{G}(t;\alpha) = \widetilde{\mathcal{F}}(t;\alpha,0)$. Then, for $\mathcal{U}(\lambda,\mu)$ of the form $\mathcal{U}(\lambda,\mu) = \mathcal{V}(\lambda-\mu)$, (46) may be written:

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{G} = \widetilde{\mathcal{L}}\mathcal{G} \\ \mathcal{G}(0;.,.) \equiv 0 \end{cases}$$

where

$$\widetilde{\mathcal{L}}(t,\alpha) = \sum_{j=-\infty}^{\infty}\alpha_j\frac{\partial}{\partial\alpha_j} + 2\sum_{jk}F_{j+k}\left(\frac{\alpha}{\sqrt{2}},\frac{\alpha}{\sqrt{2}}\right)\frac{\partial^2}{\partial\alpha_j\partial\alpha_k}.$$

Exactly the same arguments as before give that $\widetilde{\mathcal{L}}$ is the infinitesimal generator of a Feller semigroup, from which it follows that $\mathcal{G}(t,.) \equiv 0$ for all $t > 0$. $\square$

Establishing the Result The remainder is now straightforward. Consider the function

$$\mathcal{U}(\lambda,\mu) = 1 - \exp\left\{-\sum_{n=0}^{\infty}e^{-n}f((\lambda_n-\mu_n)(\lambda_{-n}-\mu_{-n}))\right\}$$

where $f:\mathbb{R}_+\to[0,2]$ is a non-decreasing function satisfying $f(x) = x$ for $x\in[0,1]$, $\lim_{x\to+\infty}f(x) = 2$ and $f'(x)\le 1$, $|f''(x)| < 2$. Such a choice of $\mathcal{U}$ satisfies (38) and (40). Let

$$\mathcal{F}(t;\lambda,\mu) = \mathbb{E}_{(\lambda,\mu)}\left[\mathcal{U}(\lambda(t),\mu(t))\right].$$

From the above argument, it follows that $\widetilde{\mathcal{F}}(t;\alpha,0) = 0$ for all $t\ge 0$. and hence that, for each n, $|\lambda_n(t)-\mu_n(t)| = 0$, $\mathbb{Q}$ almost surely, for Lebesgue-almost all $t > 0$. Since $\|u-v\|_2^2(t) = \sum_n|\lambda_n(t)-\mu_n(t)|^2$, it follows that for almost all $t > 0$ and all $0\le N < +\infty$, $\mathbb{E}\left[N\wedge\|u-v\|_2^{2\gamma}(t)\right] \equiv 0$ for Lebesgue almost all $t\ge 0$.

Let $U(t) = \int_{\mathbb{S}^1}u(t,x)dx$ and $V(t) = \int_{\mathbb{S}^1}v(t,x)dx$, then $U-V$ is a *continuous* local martingale. Furthermore, $|U(t)-V(t)| \le \left(\int_{\mathbb{S}^1}(u(t,x)-v(t,x))^2dx\right)^{1/2}$ so that, $\mathbb{Q}$ almost surely, it follows that

$U(t)-V(t)=0$ for Lebesgue almost all $t\geq 0$. Using continuity of $V(t)-U(t)$, it follows that $\mathbb{Q}$-almost surely, $\sup_{0\leq t\leq T}|U(t)-V(t)|=0$ for any fixed $T<+\infty$. Using the fact that $(U(t)-V(t))^2-\int_0^t\int_{\mathbb{S}^1}(u^\gamma(s,x)-v^\gamma(s,x))^2dxds$ is a *continuous* local martingale, it follows that

$$\lim_{t\to+\infty}\int_0^t\int_{\mathbb{S}^1}(u^\gamma(s,x)-v^\gamma(s,x))^2dxds=0 \qquad \mathbb{Q}-\text{almost surely.}$$

Together with the a-priori bound

$$\mathbb{E}\left[\left(\int_0^\infty\int_{\mathbb{S}^1}(u^\gamma(s,x)-v^\gamma(s,x))^2dxds\right)^{\alpha/2}\right]\leq 2^{1+(\alpha/2)}\widetilde{K}(\alpha) \qquad \alpha\in(0,1)$$

for a universal constant $\widetilde{K}(\alpha)<+\infty$ for $\alpha\in(0,1)$, depending only on α, gives that for $0<\alpha<1$:

$$d_{2\gamma,\alpha}(u,v)\leq\mathbb{E}\left[\left(\int_0^\infty\int_{\mathbb{S}^1}(u^\gamma(s,x)-v^\gamma(s,x))^2dxds\right)^{\alpha/2}\right]=0,$$

thus completing the proof of Theorem 5.1. $\square$

6 Existence of norms

Let u denote a solution to Equation (1). In this section, the following result is proved.

Theorem 6.1. *Let u denote a solution in $\mathcal{S}_{2\gamma,\alpha}$ for $\alpha<1$ to Equation (1). Let*

$$\|u\|_p(t)=\left(\int_{\mathbb{S}^1}u(t,x)^pdx\right)^{1/p}.$$

Then for each $p<+\infty$ and each $\alpha\in\left(0,\frac{1}{2}\right)$ and each $T<+\infty$ such that the initial condition u_0 satisfies $\int_0^T\|P_tu_0\|_{2p}^\alpha dt<+\infty$, there is a constant $C(p,\alpha,T,u_0)<+\infty$ such that

$$\mathbb{E}\left[\int_0^T\|u\|_{2p}^\alpha(t)dt\right]<C(p,\alpha,T,u_0).$$

Proof of Theorem 6.1 Let

$$U(s,t;x)=P_tu_0(x)+\int_0^s\int_{\mathbb{S}^1}p_{t-r}(x-y)u^\gamma(r,y)W(dy,dr).$$

Then $u(t,x)=U(t,t;x)$. By Itô's formula,

$$\begin{aligned}U(s,t;x)^{2p} &= (P_tu_0(x))^{2p}+2p\int_0^s\int_{\mathbb{S}^1}\left(U(r,t;x)^{2p-1}p_{t-r}(x-y)\right)u^\gamma(r,y)W(dy,dr)\\ &\quad +p(2p-1)\int_0^s\int_{\mathbb{S}^1}\left(U(r,t;x)^{2p-2}p_{t-r}^2(x-y)\right)u^{2\gamma}(r,y)dydr.\end{aligned}$$

Let $\|U(s,t;x)\|_p=\left(\int_{\mathbb{S}^1}U(s,t;x)^pdx\right)^{1/p}$. Then, using $\int_{\mathbb{S}^1}p_{t-r}^{2p}(x-y)dx\leq\left(1+\frac{c(p)}{(t-r)^{p-(1/2)}}\right)$ for some $c(p)$ and Hölder's inequality,

$$
\begin{aligned}
\|U(s,t)\|_{2p}^{2p} \quad \leq \quad & \|P_t u_0\|_{2p}^{2p} + 2p \int_0^s \int_{\mathbb{S}^1} \left(\int_{\mathbb{S}^1} U(r,t;x)^{2p-1} p_{t-r}(x-y) dx \right) u^\gamma(r,y) W(dy,dr) \\
& + p(2p-1) \int_0^s \left(1 + \frac{c(p)}{(t-r)^{1-(1/2p)}} \right) \|U(r,t)\|_{2p}^{2p-2} \|u(r)\|_{2\gamma}^{2\gamma} dr.
\end{aligned}
$$

It follows, again by Itô's formula, that

$$
\begin{aligned}
& \|U(s,t)\|_{2p}^{2pq} \leq \|P_t u_0\|_{2p}^{2pq} \\
& \quad + 2pq \int_0^s \|U(r,t)\|_{2p}^{2p(q-1)} \int_{\mathbb{S}^1} \left(\int_{\mathbb{S}^1} U(r,t;x)^{2p-1} p_{t-r}(x-y) dx \right) u^\gamma(r,y) W(dy,dr) \\
& \quad + p(2p-1) q \int_0^s \left(1 + \frac{c(p)}{(t-r)^{1-(1/2p)}} \right) \|U(r,t)\|_{2p}^{2pq-2} \|u(r)\|_{2\gamma}^{2\gamma} dr \\
& \quad + 2p^2 q(q-1) \int_0^s \|U(r,t)\|_{2p}^{2p(q-2)} \left(\int_{\mathbb{S}^1} U(r,t;x)^{2p-1} p_{t-r}(x-y) dx \right)^2 \|u(r)\|_{2\gamma}^{2\gamma} dr.
\end{aligned}
$$

For $0 < q < 1$, the last term is negative and so may be disregarded for obtaining an upper bound. It follows by the Burkholder-Davis-Gundy inequality, that for $\alpha \in \left(0, \frac{1}{2}\right)$ and $q \in (0,1)$, there are constants $c(\alpha,p,q)$ and $c(p)$ such that

$$
\begin{aligned}
\mathbb{E}\left[\|u(t)\|_{2p}^{2pq\alpha} \right] \quad \leq \quad & \|P_t u_0\|_{2p}^{2pq\alpha} \\
& + c(\alpha,p,q) \mathbb{E}\left[\left(\int_0^t \left(1 + \frac{c(p)}{(t-r)^{1-(1/2p)}} \right) \|U(r,t)\|_{2p}^{4p(q-1)+4p-2} \|u(r)\|_{2\gamma}^{2\gamma} dr \right)^{\alpha/2} \right] \\
& + c(\alpha,p,q) \mathbb{E}\left[\left(\int_0^s \left(1 + \frac{c(p)}{(t-r)^{1-(1/2p)}} \right) \|U(r,t)\|_{2p}^{2pq-2} \|u(r)\|_{2\gamma}^{2\gamma} dr \right)^{\alpha} \right].
\end{aligned}
$$

Firstly, by Jensen's inequality, for a non-negative function f and $\beta \in (0,1)$,

$$
\int_0^T f(s)^\beta ds \leq T^{1-\beta} \left(\int_0^T f(s) ds \right)^\beta
$$

and, for $r \in [0,T]$, $\int_r^T \frac{1}{(t-r)^{1-(1/2p)}} dr \leq 2pT^{1/2p}$. Note that for $2p \geq 1$, $\|U(r,t)\|_{2p} \geq U(r)$, from which it follows, with $q = \frac{1}{2p}$ and $T < +\infty$ that there is a constant $c(\alpha,p,T) < +\infty$ such that

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T \|u(t)\|_{2p}^\alpha dt \right] \quad \leq \quad & \int_0^T \|P_t u_0\|_{2p}^\alpha dt \\
& + c(\alpha,p,T) \left(\mathbb{E}\left[\left(\int_0^T \|u(r)\|_{2\gamma}^{2\gamma} dr \right)^{\alpha/2} \right] + \mathbb{E}\left[\left(\int_0^T \frac{1}{U(r)} \|u(r)\|_{2\gamma}^{2\gamma} dr \right)^{\alpha} \right] \right).
\end{aligned}
$$

By Itô's formula,

$$
U(t) \log U(t) + U(t) = 1 + \int_0^t (2 + \log U(s)) dU(s) + \frac{1}{2} \int_0^t \frac{1}{U(s)} \|u\|_{2\gamma}^{2\gamma}(s) ds
$$

so that for $\alpha < \frac{1}{2}$, using Hölder's inequality, there is a $c(\alpha) < +\infty$ such that

$$
\begin{aligned}
&\frac{1}{2^\alpha}\mathbb{E}\left[\left(\int_0^T \frac{1}{U(r)}\|u\|_{2\gamma}^{2\gamma}(r)dr\right)^\alpha\right] \\
&\qquad \leq 1+\mathbb{E}[|U(T)\log U(T)|^\alpha]+\mathbb{E}[U(T)^\alpha]+c(\alpha)\mathbb{E}\left[\left(\int_0^T (2+\log U(s))^2\|u\|_{2\gamma}^{2\gamma}(s)ds\right)^{\alpha/2}\right]
\end{aligned}
$$

Again, by Itô's formula,

$$
\begin{aligned}
&\frac{15}{4}U(t)^2-\frac{3}{2}U(t)^2\log U(t)+\frac{1}{2}U(t)^2(\log U(t))^2 \\
&\qquad =\frac{15}{4}+\int_0^t (6U(s)-2U(s)\log U(s)+U(s)(\log U(s))^2)dU(s)+\frac{1}{2}\int_0^t (2+\log U(s))^2\|u\|_{2\gamma}^{2\gamma}(s)ds
\end{aligned}
$$

giving, for $\alpha \in \left(0,\frac{1}{2}\right)$,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t (2+\log U(s))^2\|u\|_{2\gamma}^{2\gamma}(s)ds\right)^\alpha\right] \leq \left(\frac{15}{2}\right)^\alpha (1+\mathbb{E}[U(t)^{2\alpha}])+3^\alpha\mathbb{E}[U(t)^{2\alpha}|\log U(t)|^\alpha] \\
&\quad +\mathbb{E}[U(t)^{2\alpha}(\log U(t))^{2\alpha}]+c(\alpha)\mathbb{E}\left[\left(\int_0^t (6U(s)-2U(s)\log U(s)+U(s)(\log U(s))^2)^2\|u\|_{2\gamma}^{2\gamma}(s)\right)^{\alpha/2}\right] \\
&\quad \leq \left(\frac{15}{2}\right)^\alpha (1+\mathbb{E}[(\sup_t U(t))^{2\alpha}])+3^\alpha\mathbb{E}[(\sup_t U(t)|\log U(t)|^{1/2})^{2\alpha}]+\mathbb{E}[\sup_t (U(t)|\log U(t)|)^{2\alpha}] \\
&\quad +c(\alpha)\mathbb{E}\left[\left(\sup_t (6U(t)+2U(t)|\log U(t)|+U(t)(\log U(t))^2)\right)^{2\alpha}\right]^{1/2}\mathbb{E}\left[\left(\int_0^\infty \|u\|_{2\gamma}^{2\gamma}(s)ds\right)^\alpha\right]^{1/2} \\
&\quad < +\infty.
\end{aligned}
$$

Theorem 6.1 follows. ☐

7 Conclusion and Further Study

In this article, existence and uniqueness of solution to Equation (1) in appropriate spaces was established, thus answering the question posed in Mueller [7], of whether the solution could be continued after explosion of the L^∞ norm.

The main outstanding question remaining is the nature of the explosions in the L^∞ space norm. The results taken together; that $\sup_t U(t) < +\infty$ where U is the total mass process, that $\int_0^\infty \int_{\mathbb{S}^1} u^{2\gamma}(t,x)dxdt < +\infty$ and the results about L^p spatial norms in the final section should give clear limitations on the nature of the explosions (or sizes of the spikes) that can occur. It would be interesting to have more detailed information about the behaviour of the solution close to explosion points.

More generally, the existence and uniquess results established in this article, while restricted to a power $\gamma > 1$, indicates that there are well defined solutions for potential terms of arbitrary polynomial growth, which are Lipschitz at 0 (the techniques for existence rely on non-negativity of solution, which requires assumptions on the potential in a neighbourhood of 0; uniqueness requires locally Lipschitz

in a neighbourhood of 0). The noise coefficient only requires to be locally Lipschitz. There is the open problem of establishing a machinery for the study of SPDEs which reflects this; machinery which requires a global Lipschitz assumption in order to prove existence and uniquess by applying a Gronwall lemma misses the essential nature of the process.

The subject of partial differential equations is largely motivated by the natural and engineering sciences and largely seeks to answer problems raised within these disciplines. The same is true of the subject of SPDEs and good examples may be found, for example, in Walsh [12]. While the particular SPDE addressed in this article presents a problem that is of interest in its own right, it would also be of interest to consider situations from applied fields which motivate its study. The SPDE would then be considered as the limit, at least formally, of a sequence of approximating equations indexed by a parameter ϵ, the limiting equation occurring as $\epsilon \to 0$. The comparison of behaviour between the 'physical' equations with $\epsilon > 0$ and the limit, for example how explosions develop in the limit, is of interest.

This article therefore answers one question, but by showing global existence and uniqueness for potential u^γ for $\gamma > 1$, indicates that there is a rather large field that has substantial potential for further development.

Acknowledgements The author acknowledges invaluable contributions from two anonymous referees which led to substantial improvements and thanks them for their very careful reading of the article. In addition, the author acknowledges invaluable discussions with Carl Mueller.

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